Inference for Stochastic Processes
4. Lévy Processes

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Poisson Process

- $\tau_n = \sum_1^n \delta_j$, $\delta_j \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$
- $X_t \equiv \sup\{n \in \mathbb{Z}_+ : t \geq \tau_n\}$, $0 \leq t < \infty$
- $[X_{t_{j+1}} - X_{t_j}] \overset{\text{ind}}{\sim} \mathsf{Po}\big([t_{j+1} - t_j]\big), \quad 0 \leq t_0 \leq t_1 \leq \ldots < \infty$
- $\mathsf{E}[e^{i\omega X_t}] = \sum_{k=0}^\infty e^{i\omega k} \frac{t^k\, e^{-t}}{k!} = e^{t(e^{i\omega}-1)}$
- $\mathsf{E}[e^{i\omega(u\, X_{\lambda t})}] = \sum_{k=0}^\infty e^{i\omega k u} \frac{(\lambda t)^k\, e^{-\lambda t}}{k!} = e^{t(e^{i\omega u}-1)\lambda}$

Simple Compound Poisson Process

- $Y_t = \sum u_j X_j(t\lambda_j)$, $u_j \in \mathbb{R}$, $\lambda_j > 0$
- Setting $\nu(du) \equiv \sum \lambda_j \delta_{u_j}(du)$,

$$\mathsf{E}[e^{i\omega Y_t}] = e^{t \sum_j (e^{i\omega u_j}-1)\lambda_j} = e^{t\int (e^{i\omega u}-1)\,\nu(du)}$$

Properties:

- Y_t is a Markov process with Stationary Independent Increments
- $\mathsf{E}[Y_t - Y_s] = (t-s)\int u\nu(du)$
- $\mathsf{V}[Y_t - Y_s] = (t-s)\int u^2\nu(du)$

Limits of Simple Compound Poisson Processes

If a sequence of Simple Compound Poisson Processes $Y_t^{(n)}$ have their Characteristic Functions (ch.f.'s) converge pointwise

$$\begin{aligned} \mathsf{E}[e^{i\omega Y_t^{(n)}}] &= \exp\left\{t\int \left[e^{i\omega u}-1\right]\nu^{(n)}(du)\right\} \\ &\to \exp\left\{t\int \left[e^{i\omega u}-1\right]\nu(du)\right\} \end{aligned}$$

to a continuous function then the processes converge in distribution too. What sort of limits can we have? Evidently $\nu(du)$ must satisfy

$$\int_{\mathbb{R}} \left|e^{i\omega u}-1\right|\nu(du) < \infty$$

Since $|e^{i\omega u}-1| \leq 2 < \infty$, this is satisfied by any finite measure.

General Compound Poisson Processes

The simplest case is if $\nu(\mathbb{R}) < \infty$ to guarantee that

$$\int_{\mathbb{R}} \left| e^{i\omega u} - 1 \right| \nu(du) < \infty.$$

In this case Y_t has piecewise-constant paths with independent jumps at discrete times $0 < \tau_1 < \cdots < \infty$; the holding-times $[\tau_n - \tau_{n-1}] \sim \mathsf{Ex}\big(\nu(\mathbb{R})\big)$ have independent exponential distributions with means $\mathsf{E}[\tau_n - \tau_{n-1}] = \frac{1}{\nu(\mathbb{R})}$, while the jumps have distributions

$$\left[Y_{\tau_n} - Y_{\tau_n -}\right] \sim \frac{\nu(du)}{\nu(\mathbb{R})}.$$

The process Y_t will only be L^1 if this jump distribution is L^1.

Examples

- $\nu(du) = \lambda \delta_c(du)$: Rescaled Poisson,
$$Y_t = cX_{\lambda t}$$
- $\nu(du) = \delta_{-1}(du) + \delta_1(du)$: Symmetric Poisson,
$$Y_t = X_t^{(1)} - X_t^{(2)}$$
- $\nu(du) = \frac{\lambda e^{-u^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, du$: Poisson sum of normals,
$$Y_t = \sum_{j=1}^{X_{\lambda t}} \sigma Z_j$$
- $\nu(du) = \alpha\beta e^{-\beta u}\, du$, $u > 0$: Poisson sum of exponentials,
$$Y_t = \sum_{j=1}^{X_{\alpha t}} \delta_j / \beta$$

Finite-Variation Limits

The mean number of jumps per unit time is $\nu(\mathbb{R})$; the process could still make sense with infinitely many "small" jumps, so long as they are always summable. We still need to have only finitely many "big" jumps per unit time, so for every $\epsilon > 0$ we'll need

$$\nu\big((-\epsilon, \epsilon)^c\big) < \infty$$

(otherwise the sum of all the jumps would be indeterminate), but it would be okay to have $\nu(\mathbb{R}) = \infty$ so long as $\int_{-\epsilon}^{\epsilon} |u| \nu(du) < \infty$. These two conditions can be combined into the single requirement

$$\int_{\mathbb{R}} \left(|u| \wedge 1\right) \nu(du) < \infty,$$

which, since $[e^{i\omega u} - 1] = i\omega u + o(u)$ near $u \approx 0$, will still ensure that

$$\int_{\mathbb{R}} \left| e^{i\omega u} - 1 \right| \nu(du) < \infty.$$

Examples

- $\nu(du) = u^{-1} e^{-u}\, du$, $u > 0$: Gamma Process, $Y_t \sim \mathsf{Ga}(t, 1)$ Rate of jumps bigger than ϵ is $\nu\big((-\epsilon, \epsilon)^c\big) = \int_\epsilon^\infty u^{-1} e^{-u}\, du \equiv E_1(\epsilon) < \infty$, the "exponential integral function." The integrability condition is satisfied because
$$\int_{\mathbb{R}} \left(1 \wedge |u|\right) \nu(du) = \int_0^1 e^{-u}\, du + \int_1^\infty u^{-1} e^{-u}\, du < \infty$$
- $\nu(du) = \xi u^{-1-\xi}\, du$, $u > 0$: Skewed Stable Process of index ξ, $Y_t \sim \mathsf{St}(\xi, 1, 1, 0)$, with
$$\int_{\mathbb{R}} \left(1 \wedge |u|\right) \nu(du) = \int_0^1 \xi u^{-\xi}\, du + \int_1^\infty \xi u^{-1-\xi}\, du = \frac{1}{1 - \xi}$$
This is finite so long as $0 < \xi < 1$.

Finite Quadratic Variation Limits

The condition $\int_{\mathbb{R}} (1 \wedge |u|)\, \nu(du) < \infty$ is *necessary* to guarantee that the ch.f.

$$\exp\left(t \int_{\mathbb{R}} [e^{i\omega u} - 1]\, \nu(du) \right) \tag{1}$$

will exist, and for *increasing* processes Y_t (*i.e.*, those with only positive jumps) this is as far as we can go.

Still if we allow both positive and negative jumps it is possible to go a bit further. We can have so many tiny jumps that they can't be summed absolutely, with a kind of mystical "infinite cancellation"!

Here's how it works. We have already used the fact that $e^{i\omega u} = 1 + O(u)$ to ensure that (1) will be well-defined whenever $\int_{\mathbb{R}} (1 \wedge |u|)\, \nu(du) < \infty$; we can do better if we use the second-order approximation $e^{i\omega u} = 1 + i\omega u + O(u^2)$.

For any function $h(u)$ with $\int |h(u)|\nu(du) < \infty$ we could set $\mu = \int h(u)\, \nu(du)$ and rewrite the ch.f. as

$$\mathsf{E}\left[e^{i\omega Y_t}\right] = \exp\left(it\omega\mu + t \int \left(e^{i\omega u} - 1 - i\omega h(u)\right)\, \nu(du) \right) \tag{2}$$

(the "$it\omega\mu$" part corresponds to linear drift). For any function $h(u) = u + O(u^2)$ near $u \approx 0$, we have $\left[e^{i\omega u} - 1 - i\omega h(u)\right] = O(u^2)$ near $u \approx 0$, so (2) makes sense for any linear drift $\mu \in \mathbb{R}$ and any "jump measure" $\nu(du)$ satisfying

$$\int_{\mathbb{R}} (1 \wedge u^2)\, \nu(du) < \infty$$

The resulting SII process will have paths with infinite total variation if $\int_{\mathbb{R}} (1 \wedge |u|)\, \nu(du) = \infty$, but the *quadratic* variation and the process itself will still be well-defined and finite.

Examples

- $\nu(du) = |u|^{-2}\, du$: Cauchy Process, $Y_t \sim \mathsf{Ca}(0, 1)$
  Rate of jumps bigger than ϵ is $\nu\big((-\epsilon, \epsilon)^c\big) = 2\int_\epsilon^\infty u^{-2}\, du = \frac{2}{\epsilon}$
  The new integrability condition is satisfied because

  $$\int_{\mathbb{R}} (1 \wedge |u|^2)\, \nu(du) = 2\int_0^1 1\, du + 2\int_1^\infty u^{-2}\, du = 4 < \infty$$

- $\nu(du) = \xi |u|^{-1-\xi}\, du$: Symmetric Stable Process of index ξ, $Y_t \sim \mathsf{St}(\xi, 1, 0, 0)$, with

  $$\int_{\mathbb{R}} (1 \wedge |u|^2)\, \nu(du) = 2\int_0^1 \xi u^{1-\xi}\, du + 2\int_1^\infty \xi u^{-1-\xi}\, du = \frac{4}{2-\xi}$$

  This is finite so long as $0 < \xi < 2$ (note increase in range of ξ).

Compensated Compound Poisson Process

- $Y_t = \mu t + \sum u_j X(t\lambda_j) - t \sum h(u_j)\lambda_j$

$$\begin{aligned}
\mathsf{E}[e^{i\omega Y_t}] &= \exp\left(it\omega\mu + t \sum_j [e^{i\omega u_j} - 1 - i\omega h(u_j)]\lambda_j \right) \\
&= \exp\left(it\omega\mu + t \int [e^{i\omega u} - 1 - i\omega h(u)]\, \nu(du) \right), \\
&\quad \text{where } \nu(du) \equiv \sum \lambda_j \delta_{u_j}(du)
\end{aligned}$$

- Total variation: $||Y||_t = t\big|\mu - \sum h(u_j)\lambda_j\big| + \sum |u_j|$
- Quadratic variation: $[Y]_t = \sum |u_j|^2$
- Any process of form (2) can be approximated in this way.

Some Properties

- Y_t is a Markov process with Stationary Independent Increments
- $\mathsf{E}[Y_t - Y_s] = (t-s)\left[\mu + \int_{\mathbb{R}} u\nu(du)\right]$, if $\int_{\mathbb{R}} |u|\,\nu(du) < \infty$ (otherwise mean does not exist)
- $\mathsf{V}[Y_t - Y_s] = (t-s)\int_{\mathbb{R}} u^2\nu(du)$
- Set
$$\mathcal{L}\phi(x) \equiv \int_{\mathbb{R}} \left[\phi(x+u) - \phi(x) - h(u)\phi'(x)\right]\nu(du)$$
for $\phi \in \mathcal{C}_b^2(\mathbb{R})$; then
$$M_t^\phi \equiv \phi(Y_t) - \phi(Y_0) - \int_0^t \mathcal{L}\phi(Y_{s-})\,ds$$
is a martingale.

Review: Path Description

- If $\lambda = \nu(\mathbb{R}) < \infty$, paths stay constant for exponential holding times with mean $1/\lambda$, then take independent jumps with distribution $\nu(du)/\lambda$ (*e.g.*, Compound Poisson);
- If $\nu(\mathbb{R}) = \infty$ but $\int |u|\,\nu(du) < \infty$, paths have infinitely many jumps in every time interval, but total variation and process itself have finite means (*e.g.*, Gamma);
- If $\int |u|\,\nu(du) = \infty$ but $\int(|u| \wedge 1)\,\nu(du) < \infty$, total variation is still finite but now has infinite expectation, and process mean is infinite or undefined (*e.g.*, Stable, $0 < \xi < 1$);
- If $\int(|u| \wedge 1)\,\nu(du) = \infty$, but $\int(|u|^2 \wedge 1)\,\nu(du) < \infty$, paths have infinite total variation, but finite *quadratic* variation (*e.g.*, Stable, $1 \le \xi < 2$).

Lévy Processes (Lévy-Khinchine Theorem)

a **Every** process with **stationary independent increments** (SII) has ch.f. of the form

$$E\left[e^{i\omega X_t}\right] = \exp\left(it\omega\mu - \frac{t\omega^2\sigma^2}{2} + t\int \left(e^{i\omega u} - 1 - i\omega h(u)\right)\,\nu(du)\right)$$

with $\int_{\mathbb{R}} \left(1 \wedge |u|^2\right)\nu(du) < \infty$, where $h(u)$ is an arbitrary bounded continuous function with $h(u) = u + O(u^2)$ near $u \approx 0$.

Thus every SII process can be decomposed as Brownian Motion with Drift plus a Lévy jump process,

$$X_t = \mu t + \sigma\omega_t + Y_t.$$

The “compensator” $h(u)$ is unnecessary if $\int_{\mathbb{R}} \left(1 \wedge |u|\right)\nu(du) < \infty$. The measure $\nu(du)$ and diffusion σ^2 are determined uniquely, but the choices of μ and $h(u)$ are intertwined.

ILM Construction

Let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and let $\nu(du)$ be a measure satisfying $\int_{\mathbb{R}} \left(1 \wedge |u|\right)\nu(du) < \infty$. Let $H(du, ds) \sim \mathsf{Po}\big(\nu(du) \times ds\big)$ be a Poisson random field on $\mathbb{R} \times \mathbb{R}_+$, and let ω_t be a Wiener process. Then

$$Y_t = \mu t + \sigma\omega_t + \iint_{\mathbb{R}\times[0,t]} u\,H(du\,ds)$$

is a Lévy process with ch.f.

$$E\left[e^{i\omega Y_t}\right] = \exp\left(it\omega\mu - \frac{t\omega^2\sigma^2}{2} + t\int \left(e^{i\omega u} - 1\right)\,\nu(du)\right)$$

ILM Construction (cont'd)

Here's how to generate the needed Poisson random field $H(du, ds) \sim \mathsf{Po}\big(\nu(du) \times ds\big)$:

Begin by picking a small number $\epsilon > 0$, any $t > 0$, and initialize $n = 0$ and $\tau_0 = 0$. Then:

1. Increment $n \leftarrow n + 1$ and $\tau_n = \tau_{n-1} + \delta_n$, for $\delta_n \sim \mathsf{Ex}(1)$.
2. Draw $s_n \sim \mathsf{Un}(0, t)$
3. Set $u_n \equiv \inf\{u \geq 0 : \ \nu\big((-\infty, -u] \cup [u, \infty)\big) \leq \tau_n\}$ ("Inverse Lévy Measure", or *ILM*... W&Ickstadt, Bka 1998)
4. If $u_n < \epsilon$, quit; otherwise, return to step 1. above.

Now $\{(\pm u_n, s_n)\}$ are the events of $H(du, ds)$, restricted to $(-\epsilon, \epsilon)^c \times [0, t]$. Take limit as $\epsilon \to 0$.

ILM Construction With Compensation

Let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, and let $\nu(du)$ be a measure satisfying $\int_{\mathbb{R}} \left(1 \wedge |u|^2\right) \nu(du) < \infty$. Fix a bounded function $h(u) = u + o(u^2)$ and set

$$\mu_\epsilon \equiv \mu - \int_{(-\epsilon,\epsilon)^c} h(u)\nu(du).$$

Let $H(du, ds) \sim \mathsf{Po}\big(\nu(du) \times ds\big)$ be a Poisson random field on $\mathbb{R} \times \mathbb{R}_+$, and let ω_t be a Wiener process. Then

$$Y_t = \lim_{\epsilon \searrow 0} \mu_\epsilon t + \sigma \omega_t + \iint_{(-\epsilon,\epsilon)^c \times [0,t]} u \, H(du\, ds)$$

is a Lévy process with ch.f.

$$E\left[e^{i\omega Y_t}\right] = \exp\left(it\omega\mu - \frac{t\omega^2\sigma^2}{2} + t\int \left(e^{i\omega u} - 1 - i\omega h(u)\right) \nu(du)\right)$$

Implications for Inference

If we observe Y_s for all $s \in [0, t]$, then

- We learn σ^2 exactly, from the quadratic variation of the continuous part $Y_t - \sum_{s \leq t}[Y_s - Y_{s-}]$;
- We learn jumps u_j and jump times s_j exactly; inference about $\nu(du)$ is now equivalent to inference about Poisson random field. The Likelihood Ratio based on big jumps $|u_j| > \epsilon$ is:

$$\frac{L(\nu)}{L(\eta)} = e^{-t\nu\left((-\epsilon,\epsilon)^c\right) + t\eta\left((-\epsilon,\epsilon)^c\right)} \prod_{s_j \leq t,\ |u_j| > \epsilon} \frac{d\nu}{d\eta}(u_j)$$

Example 1

If $Y_t \sim \mathsf{Ga}(\alpha t, \beta)$ with β unknown (but α known), let $\eta(du)$ correspond to $\beta = 1$ and $\nu(du)$ to arbitrary β. Then

$$\nu\left[(-\epsilon, \epsilon)^c\right] = \int_\epsilon^\infty \alpha u^{-1} e^{-\beta u}\, du = \alpha E_1(\epsilon\beta) \approx c - \alpha \log \epsilon\beta$$

and

$$\frac{\nu(du)}{\eta(du)} = \frac{\alpha u^{-1} e^{-\beta u}}{\alpha u^{-1} e^{-u}} = e^{(1-\beta)u},$$

so

$$\begin{aligned} \frac{L(\nu)}{L(\eta)} &= e^{\alpha t[E_1(\epsilon) - E_1(\beta\epsilon)]} \prod_{s_j \leq t,\ |u_j| > \epsilon} e^{(1-\beta)u_j} \\ &= \exp\left(\alpha t[E_1(\epsilon) - E_1(\epsilon\beta)] + (1-\beta)\sum u_j\right) \qquad (3) \\ &\to \exp\left(\alpha t \log\beta + (1-\beta)Y_t\right), \text{ so } \hat{\beta} = \alpha t / Y_t. \end{aligned}$$

Perspective

If we had observed $Y_s \sim \mathsf{Ga}(\alpha s, \beta)$ with β unknown (but α known) only at times $s_j = j\frac{t}{n}$, $j = 1:n$, the likelihood would depend only on the independent increments $\Delta_j \sim \mathsf{Ga}(\alpha_n, \beta)$:

$$\begin{aligned}\frac{L(\nu)}{L(\eta)} &= \frac{\prod \beta^{\frac{\alpha t}{n}} (\Delta_j)^{\frac{\alpha t}{n}-1} e^{-\beta\Delta_j} / \Gamma(\frac{\alpha t}{n})}{\prod (\Delta_j)^{\frac{\alpha t}{n}-1} e^{-\Delta_j} / \Gamma(\frac{\alpha t}{n})} \\ &= \beta^{n\frac{\alpha t}{n}} e^{(1-\beta 1)\sum_j \Delta_j} \qquad (4) \\ &= \beta^{\alpha t} e^{(1-\beta)Y_t},\end{aligned}$$

just as before, so the inference would be identical— not surprising, since by (3,4) the sum $Y_t = \sum u_j = \sum \Delta_j$ is sufficient.

Example 2

If $Y_t \sim \mathsf{Ga}(\alpha t, \beta)$ with β **known** and α now unknown, let $\eta(du)$ correspond to $\alpha = 1$ and $\nu(du)$ to arbitrary α. Then again $\nu\big[(-\epsilon, \epsilon)^c\big] = \alpha E_1(\epsilon\beta) \approx c - \alpha \log \epsilon\beta$, and now

$$\frac{\nu(du)}{\eta(du)} = \frac{\alpha u^{-1} e^{-\beta u}}{u^{-1} e^{-\beta u}} = \alpha,$$

so

$$\begin{aligned}\frac{L(\nu)}{L(\eta)} &= e^{-\alpha t E_1(\beta\epsilon) + t E_1(\beta\epsilon)} \prod_{s_j \le t,\ |u_j| > \epsilon} \alpha \\ &= \exp\left((1-\alpha) t E_1(\beta\epsilon) + N_t^\epsilon \log \alpha\right)\end{aligned}$$

where N_t^ϵ is the number of jumps of sizes $|u_j| \ge \epsilon$ at times $s_j \le t$. Inference follows from taking limits in ϵ; for example,

$$\hat{\alpha} = \lim_{\epsilon \searrow 0} \frac{N_t^\epsilon}{t \log \frac{1}{\epsilon}}$$

Perspective

If we had observed $Y_s \sim \mathsf{Ga}(\alpha s, \beta)$ with β known and α unknown only at times $s_j = j\frac{t}{n}$, $j = 1:n$, the likelihood would again depend only on the independent increments $\Delta_j \sim \mathsf{Ga}(\frac{\alpha t}{n}, \beta)$:

$$\begin{aligned}\frac{L(\nu)}{L(\eta)} &= \frac{\prod \beta^{\frac{\alpha t}{n}} (\Delta_j)^{\frac{\alpha t}{n}-1} e^{-\beta\Delta_j} / \Gamma(\frac{\alpha t}{n})}{\prod \beta^{\frac{t}{n}} (\Delta_j)^{\frac{t}{n}-1} e^{-\beta\Delta_j} / \Gamma(\frac{t}{n})} \\ &= \beta^{(\alpha-1)t} \prod (\Delta_j)^{(\alpha-1)\frac{t}{n}} \left(\frac{\Gamma(\frac{t}{n})}{\Gamma(\frac{\alpha t}{n})}\right)^n \\ &= \exp\left((\alpha-1)t[\log\beta + \frac{1}{n}\sum \log \Delta_j] - n \log \frac{\Gamma(\frac{\alpha t}{n})}{\Gamma(\frac{t}{n})}\right)\end{aligned}$$

so, for example, $\hat{\alpha} = \frac{n}{t}\psi^{-1}\big(\log\beta + \overline{\log\Delta.}\big) \approx \frac{-n/t}{\gamma + \log\beta + \overline{\log\Delta.}}$, something quite different from the Gamma Process result above.

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