INTRODUCING NORMAL MODELS

Readings: GCSR Sec 2.6, 2.8, Chapter 3.

IID observations \( Y = (Y_1, Y_2, \ldots Y_n) \)

\[ Y_i \sim N(\mu, \sigma^2) \]

unknown parameters \( \mu \) and \( \sigma^2 \).

From a Bayesian perspective, it is easier to work with the precision, \( \phi \), where \( \phi = 1/\sigma^2 \).

**Likelihood**

\[
L(\mu, \phi|Y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \phi^{1/2} \exp\left\{-\frac{1}{2} \phi (Y_i - \mu)^2 \right\}
\]

\[ \propto \phi^{n/2} \exp\left\{-\frac{1}{2} \phi \sum_{i} (Y_i - \mu)^2 \right\} \]
Likelihood

\[ L(\mu, \phi|Y) \propto \phi^{n/2} \exp\left\{ -\frac{1}{2} \phi \sum_i (Y_i - \mu)^2 \right\} \]

\[ \propto \phi^{n/2} \exp\left\{ -\frac{1}{2} \phi \sum_i \left[ (Y_i - \bar{Y}) - (\mu - \bar{Y})^2 \right] \right\} \]

\[ \propto \phi^{n/2} \exp\left\{ -\frac{1}{2} \phi \left[ \sum_i (Y_i - \bar{Y})^2 + n(\mu - \bar{Y})^2 \right] \right\} \]

\[ \propto \phi^{n/2} \exp\left\{ -\frac{1}{2} \phi s^2 (n - 1) \right\} \exp\left\{ -\frac{1}{2} \phi n(\mu - \bar{Y})^2 \right\} \]

where \( s^2 = \sum_i (Y_i - \bar{Y})^2 / (n - 1) \) is the usual sample variance.
**Prior Distributions**

Conjugate prior distribution for \((\nu, \phi)\) is Normal-Gamma.

\[
\mu | \phi \sim N(\mu_0, 1/(n_0\phi)) \\
\phi \sim \text{Gamma}(\nu_0/2, (\nu_0\phi_0)/2)
\]

\[
p(\phi) \propto \phi^{\nu_0/2-1} \exp\{-\phi\nu_0\phi_0/2\}
\]

Non-informative prior distribution (improper)

\[
p(\mu, \phi) = 1/\phi
\]

assuming prior independence of location and scale parameters, \(\mu\) is uniform on the real line, \(\log(\phi)\) is uniform on real line based on Jeffreys’ invariance principle (GCSR Sec 2.8).
Posteriors under the Non-Informative Prior

\[ p(\mu, \phi|Y) \propto L(\mu, \phi)p(\mu, \phi) \]
\[ = \phi^{n/2-1} \exp\left\{-\frac{1}{2} \phi s^2 (n - 1)\right\} \exp\left\{-\frac{1}{2} \phi n(\mu - \bar{Y})^2\right\} \]
\[ = \left\{ \phi^{n-1/2} e^{-\frac{1}{2} \phi s^2 (n-1)} \right\} \left\{ \phi^{1/2} e^{-\frac{1}{2} \phi n(\mu-\bar{Y})^2} \right\} \]

\[ \propto \text{Gamma} \left( \frac{n - 1}{2}, (n - 1) \frac{s^2}{2} \right) \text{N} \left( \bar{Y}, \frac{1}{\phi n} \right) \]
\[ = p(\phi|Y)p(\mu|\phi, Y) \]
**Marginal Distribution for $\mu|Y$**

Obtain the marginal distribution for $\mu$ by integrating out $\phi$ from the joint posterior distribution, and recognize the kernel of the distribution!

$$p(\mu|Y) \propto \int p(\mu, \phi|Y) d\phi$$

$$= \int \phi^{n/2-1} \exp\left\{-\frac{1}{2} \phi s^2(n-1) + n(\mu - \bar{Y})^2\right\}$$

This has the form of a Gamma integral with $\alpha = n - 1$ and $\beta$ equal to the mess multiplying $\phi$, 

$$p(\mu|Y) \propto \left(s^2(n-1) + n(\mu - \bar{Y})^2\right)^{(n-1+1)/2}$$

$$\propto \left(1 + \frac{1}{n-1} \frac{(\mu - \bar{Y})^2}{s^2/n}\right)^{(n-1+1)/2}$$

Student-$t_{n-1}(\bar{Y}, s^2/n)$ location $\bar{Y}$, df = n-1, scale $s^2/n$)