Problem 40, page 384

\[
f_Y(y) = \int_0^\infty \frac{1}{y} e^{-(y+x/y)} dx = -e^{-(y+x/y)} \bigg|_{x=0}^\infty = e^{-y} \text{ for } y > 0 \text{ and zero otherwise.}
\]

This is exponential with parameter 1, so it has an expected value of 1.

We will use the conditional expectation to calculate \( E[X] \). First we need to find \( f(X|Y) \).

\[
f(x|y) = \frac{(1/y)e^{-(y+x/y)}}{e^{-y}} = \frac{1}{y} e^{-(x/y)} \text{ for } x, y > 0 \text{ and zero otherwise.}
\]

This is exponential with mean \( y \), which leads us to:

\[
E[X] = E[E[X|Y]] = E[Y] = 1
\]

Finally,

\[
\]

We know that \( Y \) is exponential with mean and variance equal to 1, so \( Var[Y] = E[Y^2] - E[Y]^2 \) leads us to \( E[Y^2] = 2 \).

Thus we see that \( Cov(X,Y) = 1 \).
Problem 70, page 388

a) 
\[ P(\text{heads}) = \int_{0}^{1} p \, dp = 1/2 \]

b) 
\[ P(\text{heads}) = \int_{0}^{1} p^2 \, dp = 1/3 \]

Exercise 4, page 390

The first three terms of the Taylor series expansion of \( g \) around \( \mu \) lead us to:
\[ g(x) \approx g(\mu) + (x - \mu) \cdot g'(\mu) + (x - \mu)^2 \cdot g''(\mu)/2 \]

Taking expectations of both sides leads us to our answer:
\[ E[g(x)] \approx g(\mu) + E[(x - \mu)] \cdot g'(\mu) + E[(x - \mu)^2] \cdot g''(\mu)/2 \]
\[ = g(\mu) + \sigma^2 g''(\mu)/2 \]

Exercise 19, page 392

\[ = \text{Var}(X) - \text{Cov}[X, Y] + \text{Cov}[X, Y] - \text{Var}(Y) \]
\[ = \text{Var}(X) - \text{Var}(Y) = 0 \]

Exercise 48, page 396

\[ \phi_Y(t) = E[e^{tY}] \]
\[ = E[e^{t(aX+b)}] \]
\[ = x^a \cdot E[e^{atX}] \]
\[ = x^a \cdot \phi_X(at) \]

Exercise 54, page 397

\[ \text{Cov}(Z, Z^2) = E[(Z - \mu)(Z^2 - E(Z^2))] \]
\[ = E[Z^3 - Z^2 \mu - z(\mu^2 + \sigma^2) + \mu(\mu^2 + \sigma^2)] \]
\[ = E[Z^3] - \mu(\mu^2 + \sigma^2) - \mu(\mu^2 + \sigma^2) + \mu(\mu^2 + \sigma^2) \]
\[ = E[Z^3] - \mu(\mu^2 + \sigma^2) \]

Notice that, since the standard normal distribution is symmetric, \( E[z^3] = E[-z^3] = -E[z^3] \). The only way this could be is if \( E[z^3] = 0 \). (For every point that the function is positive to the right of zero, there is a point that is negative to the left of zero.)
Another Problem

Clearly, $E[X_1] = 1$ as stated. Now we need to know $E[X_2]$, $E[X_3]$, and $E[X_4]$. For each of the next dollars handed out, the probability it going to someone from a different class is $3/4$. This is a geometric random variable with parameter $p = 3/4$. Thus its expected value is $4/3$. Similarly, $X_3$ and $X_4$ are geometric with parameters $1/2$ and $1/4$ respectively. Thus the expected value for the number of dollars paid out is $1 + 4/3 + 2 + 4 = 25/3$.

Part two asks for the probability that $4$ is paid out (which we will call $D_4$) plus the probability that $5$ is paid (which we call $D_5$).

\[
P(D_4) = 1 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{32} \\
P(D_5) = 1 \cdot \left(\frac{1}{4}\right) \left(\frac{3}{4}\right) \cdot \frac{1}{2} \cdot \frac{1}{4} \\
+ 1 \cdot \frac{3}{4} \cdot \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \cdot \frac{1}{4} \\
+ 1 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \left(\frac{3}{4}\right) \left(\frac{1}{4}\right) \\
= \frac{3}{128} + \frac{3}{64} + \frac{3}{128} = \frac{3}{32}
\]