9.0 Lesson Plan

- Discuss Quizzes/Answer Questions
- History Note
- Review
- Permutations and Combinations
- Binomial Probability
Pascal and Fermat laid out the basic rules of probability in a series of letters. One of the most famous consequences of that discussion was Pascal’s Wager.

From Kolmogorov’s Axioms, it is easy to see that for any event \( A \), \( A \) is the complement of \( A \), which is the event that \( A \) does not happen.

\[
[A] \cdot [\overline{A} | p] = 1 - [A | p]
\]

Pascal’s Wager defined two events:

\[
\{ \text{God does not exist} \} = \overline{A}
\]

\[
\{ \text{God exists} \} = A
\]

One of the most famous consequences of that discussion was Pascal’s Wager.
Pascal also defined a payoff matrix:

<table>
<thead>
<tr>
<th>( A ) True</th>
<th>( A ) False</th>
<th>( A ) True</th>
<th>( A ) False</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bet on ( A )</td>
<td>Bet on ( A )</td>
<td>( -\infty ) Reward</td>
<td>( -\infty ) Reward</td>
</tr>
<tr>
<td>( -\infty ) Loss</td>
<td>( -\infty ) Loss</td>
<td>( -\infty ) Loss</td>
<td>( -\infty ) Loss</td>
</tr>
</tbody>
</table>

Pascal argued that in terms of gambling, the optimal strategy for this game is to believe in God, no matter how small the (Bayesian subjective) value for \( P[A] \) might be.

Modern game theory does not quite agree with Pascal, but the issues are rather technical; e.g., infinite payoffs are problematic.
Review

Recall Kolmogorov’s Axioms:

\[ P[A] = \text{some possible event happens} = 1 \text{ (one of the possible outcomes must occur)} \]

\[ [B]d + [\neg A]d = [B \text{ or } \neg A]d \]

Also recall the rules of conditional probability:

\[ \frac{[B]d}{[B \text{ and } \neg A]d} = [B | \neg A]d \]

From this, note that \([A]d - 1 = [\neg A]d \). Why?

Recall Kolmogorov’s Axioms:
Consider one draw from a deck of cards. Let $A = \{\text{a king}\}$ and $B = \{\text{a queen}\}$. What is $P(A \text{ or } B)$?

These are mutually exclusive events, so

$$P(A \text{ or } B) = P(A) + P(B) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}.$$ 

Consider two draws from a deck, without replacement. Let $A = \{\text{a queen}\}$ and $B = \{\text{a king}\}$.

If $B$ and $A$ are independent events, then

$$P(A \text{ and } B) = P(A) \cdot P(B) = \frac{4}{52} \cdot \frac{4}{52} = \frac{16}{2704} = \frac{1}{171}.$$ 

Consider one draw from a deck of cards. Let $A = \{\text{a queen}\}$ and $B = \{\text{a king}\}$.
Recall that $A$ and $B$ are independent if $P[A \cap B] = P[A]P[B]$. This implies that $P[B | A] = P[A]$, why?

It also implies that $A$ and $B$ are independent if $P[A] = P[A | B]$.

This implies that $A$ and $B$ are independent if $[B]A * [A] = [A | B]B$. Why?

What is the probability of rolling a 1, then a 2, then a 3 on consecutive rolls of a fair die? Since rolls are independent, $P[1 \text{ on first \& 2 on second \& 3 on third}] = P[1 \text{ on first}]P[2 \text{ on second}]P[3 \text{ on third}] = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$. Why?
What is the probability of rolling a 1, 2, and 3 in any order in three consecutive rolls of a fair die?

Indepedence and the fact that the die is fair, the fact that different orders are mutually exclusive, and the third uses the first step lists all six possible orders; the second step uses the independence and the fact that the die is fair.

\[
\left( \frac{9}{1} \times \frac{9}{1} \times \frac{9}{1} \right) \times 6 = \\
[ (1, 2, 3) ] d + \cdots + [(3, 2, 1)] d + [(3, 1, 2)] d = \\
[(3, 1, 2) \text{ or } (1, 3, 2) \text{ or } (1, 2, 3)] d = \text{ [ in any order ] p } 1, 2, 3 \text{ in any order}
\]

Which is just \( \frac{1}{36} \).
Another way to get the same answer is to note that

\[
\frac{6}{1} \times \frac{6}{2} \times \frac{6}{3}
\]

\[= \left[ \text{one of remaining two on second roll } \right] \cdot \left[ \text{result of first two rolls } \right] \cdot \left[ \text{result of first roll } \right]
\]

\[\times \left[ \text{one of remaining two on second } \right] \cdot \left[ \text{result of first roll } \right] \cdot \left[ \text{result of roll } \right]
\]

and this is also \(1/36\).

Note that in order to get this solution, one had to be able to count all the

are called permutations.

These different arrangements ways to rearrange the numbers (1, 2, 3).
Another useful trick is to use complementary events. For example, what is the probability of getting at least one 6 in three consecutive rolls?

\[
\frac{216}{16} - 1 = \frac{9}{2} \times \frac{9}{2} \times \frac{9}{2} - 1
\]

\[
= \left[ \text{no 6 on first, no 6 on second, no 6 on third} \right]_d 
\times \left[ \text{no 6 on first, no 6 on second, no 6 on third} \right]_d 
\times \left[ \text{no 6 on first, no 6 on second, no 6 on third} \right]_d 
- 1
\]

\[
= \left[ \text{no 6 in three rolls} \right]_d 
- 1
\]

\[
= \left[ \text{at least one 6 in three rolls} \right]_d
\]
The number of ways to arrange distinct objects in a line is called the number of permutations of \( n \) distinct objects. Here \( n! \) is called the factorial and it is the number of permutations of \( n \) objects:

\[
 n! = n \times (n - 1) \times (n - 2) \times \cdots \times 1.
\]

Permutations and Combinations
By convention, $0! = 1$. For the other positive integers, $1! = 1$; $2! = 2$; $3! = 6$; $4! = 24$; $\ldots$ 

In how many ways can 8 people line up? $8! = 40320$. 

In how many ways can 4 married couples stand in a police line-up, if couples must stand together? There are $4! = 24$ ways that the couples can be arranged, and each couple can be arranged in $2! = 2$ ways. So the answer is $(4!) \times (2!) \times (2!) \times (2!) = 384$. 

\[
\frac{8!}{2!} = 40320.
\]
Thenumberofwaystoarrangeredmarblesand
bluemarblesinalineis:

\[ \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!} \]

For example, the number of ways to arrange two red marbles and two green marbles is:

\[ \binom{i(j-u)}{i/u} = \binom{j}{u} \]

The number of ways to arrange \( j \) red marbles and \( u \) blue marbles is:
This is about 212 years.

$$\binom{20}{7} = \frac{20!}{7!13!} = \left( \begin{array}{c} 20 \\ 7 \end{array} \right)$$

Why? From a set of \( n \) distinct objects, the number of ways to pick \( r \) objects is also the number of ways to pick \( r \) objects with replacement.

$$\binom{20}{7} = \frac{20 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{7 \times 7 \times 7 \times 7 \times 7 \times 7} = \left( \begin{array}{c} 7 \\ 4 \end{array} \right)$$

Note: This is just
The binomial formula gives the probability of exactly $r$ successes in $n$ trials, where each trial has the same probability of success and each trial is independent.

$$\binom{n}{r} p^r (1-p)^{n-r}$$
1. How many arrangements are there that give \( r \) heads in \( n \) tries? From the previous section, we know the answer is

\[
\binom{n}{r}
\]

2. Each arrangement is incompatible with the other arrangements.

\[
\binom{d - 1}{r - u} d = (d - 1) d \cdot \binom{d - 1}{r - u} d
\]

The second arrangement has probability \( \binom{d - 1}{r - u} d \cdot p \cdot (1 - p) \). To see this,

\[
\binom{d - 1}{r - u} d = (d - 1) d \cdot \binom{d - 1}{r - u} d
\]

Thus the probability of exactly \( r \) successes is the sum over all possible arrangements, and there are

\[
\binom{d - 1}{r - u} d \cdot p \cdot (1 - p)
\]

3. Each arrangement has the same probability.

\[
\binom{d - 1}{r - u} d \cdot p \cdot (1 - p)
\]

To consider the sequences HHTT and THHT. The first has probability

\[
\binom{d - 1}{r - u} d \cdot p \cdot (1 - p) = p^2 (1 - p)^2
\]

The second arrangement has

\[
\binom{d - 1}{r - u} d \cdot p \cdot (1 - p) = p^2 (1 - p)^2
\]

Why does this formula work?
Thus the binomial formula is

Find the probability of exactly two sixes in five rolls of a fair die.

\[
\binom{5}{2} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3 = \frac{10 
\binom{5}{2} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3 = \frac{10}{16} = 0.625
\]