MTH135/STA104: Probability

Homework # 1 Due: Tuesday, Sep 6, 2005

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Reminder: On this and every homework, give (non-integer) numberical answers as fractions *in lowest terms* or as decimals to four significant digits—so $\pi = 3.142$, $24/6 = 4$.

1. (after *p.* 30, prob. 4) Let $\Omega = \{0, 1, 2\}$ be the outcome space in a model for rolling a (six-sided) die twice and counting the total number of 1's. Tell if each of the following events can be expressed as a subset of Ω; if so, give the subset explicitly; if not, tell why:

a) Both rolls are 1's;

b) One roll is a 1, the other isn't;

c) At least one roll is a 1;

d) At least one roll is a 3;

e) The first roll is a 1.

2. The events A and B have probabilities $\mathsf{P}[A] = 0.2$ and $\mathsf{P}[B] = 0.4$, while $\mathsf{P}[A\,B] = 0.1$. Find:

a) $\mathsf{P}[A^c] =$

b) $\mathsf{P}[A \cup B] =$

c) $\mathsf{P}[A \cup B^c] =$

d) $\mathsf{P}[A\,B^c] =$

e) $\mathsf{P}[A^c\,B^c] =$

3. Find the one-dimensional integrals below, as functions of $x \ge 0$:

a) $\int_0^x e^{-c\,t}\,dt$, for each real number $c \in \mathbb{R}$;

b) $\int_{-x}^x t^2\,dt$;

c) $\int_0^x t^a\,dt$, for *every* number $a \in \mathbb{R}$

4. Let $T = \{(x, y) : 0 < x,\ 0 < y,\ 0 < x+y < 1\}$ be the indicated triangle in the first quadrant.

a) Draw the region T; label the axes.

b) Evaluate the integral

$$\iint_T 1\, dx\, dy$$

in two ways— as an iterated integral $\int \left[\int ... dx \right] dy$ (first dx, on the "inside," then dy on the "outside") and $\int \left[\int ... dy \right] dx$. Show your limits of integration clearly. Do you get the same answer both times?

c) Evaluate the integral

$$\iint_T x\, dx\, dy$$

in any way you like.

5. Evaluate the limits below:

a) $\lim_{n\to\infty} \frac{3n+4}{2n-1}$

b) $\lim_{n\to\infty} \frac{n^2+n+7}{2n^2-1}$

c) $\lim_{n\to\infty} \frac{e^n}{n}$

d) $\lim_{n\to\infty} \frac{2n+1/n}{2n+1/n}$

e) $\lim_{n\to\infty} (\pi/3)^n$

f) $\lim_{n\to\infty} (1 - 2/n)^n$

6. Find the sums, each with n terms, for each non-negative integer $n \in \mathbb{N}$:

a) $S_1 = \sum_{i=1}^{n} 3 = 3 + 3 + ... + 3$

b) $S_2 = \sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n$

c) $S_3 = \sum_{i=1}^{n} 2^i = 2 + 4 + 8 + ... + 2^n$

d) $S_4 = \sum_{i=1}^{n} 2^{-i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + 2^{-n}$

Hint: Write the sum for S_2 twice, once forward and (just below that) once backward. Compare the sums S_3 and $2 \times S_3$, again writing out the terms for one above the other. Do something similar for S_4.

7. Write as fractions in lowest terms

a) 0.45

b) 3/4 - 2/3

c) 1/99 - 1/101

8. Write as decimals to exactly four *significant* digits

a) 1/7

b) $\sqrt{200}$

c) $\sqrt{1/200}$

9. An experiment consists of testing six patients' reaction times before and after administering a drug that reproduces the effect of ingesting $1oz$ of ethanol. For each subject we record the difference D_i (before minus after) of reaction times, measured in milliseconds.

Which of the following could be an Outcome Space for this problem? Give a SHORT explanation for each answer, and for each give the number of elements $\#(\Omega)$. For a note on notation, see p. 4 below.

a) $\Omega = [0, 1]^6$, with outcomes $\omega = (D_1, D_2, ..., D_6)$;

b) $\Omega = \mathbb{R}^6 = (-\infty, \infty)^6$, with outcomes $\omega = (D_1, D_2, ..., D_6)$;

c) $\Omega = \{0, 1, 2, 3, 4, 5, 6\}$, the number of subjects whose reaction time became worse (*i.e.*, bigger);

d) $\Omega = \{1, 2, 3, 4, 5, 6\}$, one for each subject;

e) $\Omega = \{++++++, +++++-, ++++-+, ..., ------\}$, where each string of six $\pm$ signs indicates whether each of the six subjects' reaction times increased ("+") or not ("–")

10. The (arithmetic) *average* of a set A of numbers is just their sum divided by how many there are,

$$\text{avg}(A) = \sum\{a :\ a \in A\}/\#(A)$$

Set $A = \{1, 2, 3\}$. In each of the following pairs, which is larger— or are they the same?

a) $\text{avg}(\{a^2 :\ a \in A\})$ or $\text{avg}(A)^2$?

b) $\text{avg}(\{2^a :\ a \in A\})$ or $2^{\text{avg}(A)}$?

c) $\text{avg}(\{2 \times a :\ a \in A\})$ or $2 \times \text{avg}(A)$?

d) $\text{avg}(\{2 + a :\ a \in A\})$ or $2 + \text{avg}(A)$?

Note: In traditional mathematics notation, for any real numbers $a < b$, the set:

$[a, b]$ denotes the closed interval $\{x : \; a \leq x \leq b\}$;

(a, b) denotes the open interval $\{x : \; a < x < b\}$;

$\{a, b\}$ denotes the finite set containing only a and b.

For any set $\mathcal{S}$ and integer $n \in \mathbb{N}$, the symbol "$\mathcal{S}^n$" denotes the set of ordered n-tuples $(s_1, s_2, ..., s_n)$ with each $s_i \in \mathcal{S}$— for example, $\mathbb{R}^3$ is the set of three-tuples (x_1, x_2, x_3) of real numbers $x_i \in \mathbb{R} = (-\infty, \infty)$, a suitable set for indexing points in space, while $[0, 1]^6$ is the unit cube in six-dimensional space.