MTH135/STA104: Probability

Homework # 1 Solution   Due: Tuesday, Sep 6, 2005

September 16, 2005

Reminder: On this and every homework, give (non-integer) numberical answers as fractions *in lowest terms* or as decimals to four significant digits—so $\pi = 3.142$, $24/6 = 4$.

1. (after *p.* 30, prob. 4) Let $\Omega = \{0, 1, 2\}$ be the outcome space in a model for rolling a (six-sided) die twice and counting the total number of 1's. Tell if each of the following events can be expressed as a subset of Ω; if so, give the subset explicitly; if not, tell why:

a) Both rolls are 1's;

Yes, $E = \{2\}$

b) One roll is a 1, the other isn't;

Yes, $E = \{1\}$

c) At least one roll is a 1;

Yes, $E = \{1, 2\}$

d) At least one roll is a 3;

No, we could have a situation where $\{\text{first die} = 1, \text{second die} = 6\}$, which corresponds to the subset $\{1\}$ in Ω, but no 3 appeared; and have $\{\text{first die} = 3, \text{second die} = 1\}$, which again corresponds to the subset $\{1\}$ in Ω, but now one 3 appeared.

e) The first roll is a 1.

No, observe that when such event occurs, the event $\{1, 2\}$ in Ω occurs, but not vice versa. Therefore, the set $\{\text{first roll is1}\} \neq \{\text{at least in roll is1}\} = \{1\}$.

2. The events A and B have probabilities $\mathsf{P}[A] = 0.2$ and $\mathsf{P}[B] = 0.4$, while $\mathsf{P}[A\,B] = 0.1$. Find:

a) $\mathsf{P}[A^c] = 1 - \mathsf{P}[A] = 0.8$

b) $\mathsf{P}[A \cup B] = \mathsf{P}[A,] + \mathsf{P}[B] - P[A\,B] = 0.5$

c) $\mathsf{P}[A \cup B^c]$

$$\mathsf{P}[A \cup B^c] = \mathsf{P}[A] + \mathsf{P}[B^c] - P[A\,B^c] = \mathsf{P}[A] + \mathsf{P}[B^c] - (P[A] - P[A\,B]) = 0.70$$

d) $\mathsf{P}[A\,B^c] = P[A] - P[A\,B] = 0.1$

e) $\mathsf{P}[A^c\,B^c] = P[(A \cup B)^c] = 0.5$

There is no unique way to get the right answer, just make sure that the logic is correct.

3. Find the one-dimensional integrals below, as functions of $x \geq 0$:

a) $\int_0^x e^{-c\,t}\,dt$, for each real number $c \in \mathbb{R}$;

Integration by substitution, let $u = -ct$.

$$\int_0^x e^{-c\,t}\,dt = -\frac{1}{c}\int_0^x -ce^{-ct}dt = -\frac{1}{c}\int_0^{-cx} e^u du = -\frac{1}{c}(e^{-u}|_0^{-cx}) = -\frac{1}{c}(e^{-cx} - e^0) = -\frac{1}{c}(e^{-cx} - 1)$$

b) $\int_{-x}^x t^2\,dt = \frac{t^3}{3}|_{-x}^x = \frac{x^3}{3} - \frac{(-x)^3}{3} = \frac{2x^3}{3}$;

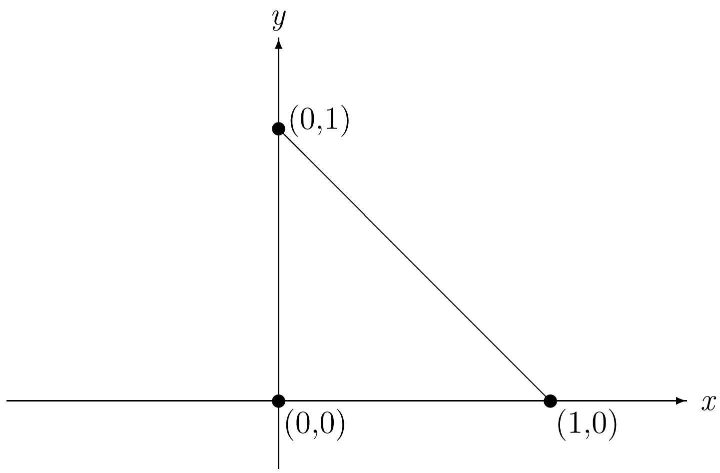
c) $\int_0^x t^a\,dt$, for *every* number $a \in \mathbb{R}$

Assume $a \neq -1$, $\int_0^x t^a\,dt = \frac{t^{a+1}}{a+1}|_0^x = \frac{x^{a+1}}{a+1}$

Assume $a = -1$, $\int_0^x t^a\,dt = \int_0^x \frac{1}{t}dt = \lim_{y\to 0}\ln(t)|_y^x = \lim_{y\to 0}\ln(x) - \ln(y) = -\infty$

4. Let $T = \{(x, y) : 0 < x,\ 0 < y,\ 0 < x+y < 1\}$ be the indicated triangle in the first quadrant.

a) Draw the region T; label the axes.



The region is inside the triangle with the edges are *not* included.

b) Evaluate the integral

$$\iint_T 1\,dx\,dy$$

in two ways— as an iterated integral $\int \left[\int ...\,dx\right] dy$ (first dx, on the "inside," then dy on the "outside") and $\int \left[\int ...\,dy\right] dx$. Show your limits of integration clearly. Do you get the same answer both times?
$\iint_T 1\,dx\,dy = \int_0^1 \int_0^{1-y} 1dxdy = \int_0^1 [x|_0^{1-y}]dy = \int_0^1 (1-y)dy = (y - \frac{y^2}{2})|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$
$\iint_T 1\,dy\,dx = \int_0^1 \int_0^{1-x} 1dydx =$ the rest is the same just exchange x and y. Works out well since there is symmetry.

c) Evaluate the integral

$$\iint_T x\,dx\,dy$$

in any way you like.
$\iint_T x\,dx\,dy = \int_0^1 \int_0^{1-y} xdxdy = \int_0^1 [\frac{x^2}{2}|_0^{1-y}]dy = \int_0^1 [\frac{1-y^2}{2} - \frac{0}{2}]dy = \int_0^1 \frac{1-y^2}{2}dy = -\frac{1}{2}\int_0^1 -(1-y)^2 dy = -\frac{1}{2}[\frac{(1-y)^2}{3}|_0^1] = -\frac{1}{2}[0 - \frac{1}{3}] = \frac{1}{6}$

5. Evaluate the limits below:

a) $\lim_{n\to\infty} \frac{3n+4}{2n-1} = \lim_{n\to\infty} \frac{3+\frac{4}{n}}{2-\frac{1}{n}} = \frac{3}{2}$

b) $\lim_{n\to\infty} \frac{n^2+n+7}{2n^2-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{7}{n^2}}{2-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n^2}} + \lim_{n\to\infty} \frac{\frac{1}{n}}{2-\frac{1}{n^2}} + \lim_{n\to\infty} \frac{\frac{7}{n^2}}{2-\frac{1}{n^2}} = \frac{1}{2} + 0 + 0 = \frac{1}{2}$

c) $\lim_{n\to\infty} \frac{e^n}{n} \geq \lim_{n\to\infty} \frac{n^2}{n} = \lim_{n\to\infty} n = \infty$

d) $\lim_{n\to\infty} \frac{2n+1/n}{2n+1/n} = \lim_{n\to\infty} 1 = 1$

e) $\lim_{n\to\infty} (\pi/3)^n \geq \lim_{n\to\infty} (1.04)^n = \infty$

f) $\lim_{n\to\infty} (1-2/n)^n = e^{-2}$

6. Find the sums, each with n terms, for each non-negative integer $n \in \mathbb{N}$:

a) $S_1 = \sum_{i=1}^{n} 3 = 3 + 3 + ... + 3 = 3n$

b) $S_2 = \sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

c) $S_3 = \sum_{i=1}^{n} 2^i = 2 + 4 + 8 + ... + 2^n = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1$

d) $S_4 = \sum_{i=1}^{n} 2^{-i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + 2^{-n} = \frac{2^{-(n+1)}-1}{\frac{1}{2}-1} = \frac{2^{-(n+1)}-1}{-\frac{1}{2}}$

Hint: Write the sum for S_2 twice, once forward and (just below that) once backward. Compare the sums S_3 and $2 \times S_3$, again writing out the terms for one above the other. Do something similar for S_4.

7. Write as fractions in lowest terms

a) $0.45 = \frac{9}{20}$

b) $3/4 - 2/3 = \frac{1}{12}$

c) $1/99 - 1/101 = \frac{2}{9999}$

8. Write as decimals to exactly four *significant* digits

a) $1/7 = 0.1429$

b) $\sqrt{200} = 14.14$

c) $\sqrt{1/200} = 0.07071$

9. An experiment consists of testing six patients' reaction times before and after administering a drug that reproduces the effect of ingesting $1oz$ of ethanol. For each subject we record the difference D_i (before minus after) of reaction times, measured in milliseconds.

Which of the following could be an Outcome Space for this problem? Give a SHORT explanation for each answer, and for each give the number of elements $\#(\Omega)$. For a note on notation, see $p.$ 5 below.

a) $\Omega = [0,1]^6$, with outcomes $\omega = (D_1, D_2, ..., D_6)$;
No. There is no guarantee that D_i would always be in the range $[0,1]$; indeed this space wouldn't LET experimental subjects have improved performance, or degraded performance beyond one ms.

b) $\Omega = \mathbb{R}^6 = (-\infty, \infty)^6$, with outcomes $\Omega = (D_1, D_2, ..., D_6)$;
Yes, this is the obvious space— just let i'th Ω_i be D_i.

c) $\Omega = \{0,1,2,3,4,5,6\}$, the number of subjects whose reaction time became worse (*i.e.*, bigger);
Yes, this is okay too– it keeps less information than b) or even e), but still has a different point for each possible outcome in a way that could conceivably help us answer the question posed in the problem

d) $\Omega = \{1,2,3,4,5,6\}$, one for each subject;
No. This would be fine for, e.g., picking one of the subjects at random— but that is NOT what happens in this experiment.

e) $\Omega = \{{+}{+}{+}{+}{+}{+}, {+}{+}{+}{+}{+}{-}, {+}{+}{+}{+}{-}{+}, ..., {-}{-}{-}{-}{-}{-}\}$, where each string of six $\pm$ signs indicates whether each of the six subjects' reaction times increased ("+") or not ("−")
Yes, this is just a more-succinct (and less informative) version of b).

10. The (arithmetic) *average* of a set A of numbers is just their sum divided by how many there are,

$$\mathrm{avg}(A) = \sum \{a : a \in A\} / \#(A)$$

Set $A = \{1,2,3\}$. In each of the following pairs, which is larger— or are they the same?

a) $\mathrm{avg}(\{a^2 : a \in A\})$ or $\mathrm{avg}(A)^2$?
$\mathrm{avg}(A)^2$

b) $\mathrm{avg}(\{2^a : a \in A\})$ or $2^{\mathrm{avg}(A)}$?
$\mathrm{avg}(\{2^a : a \in A\})$

c) $\mathrm{avg}(\{2 \times a : a \in A\})$ or $2 \times \mathrm{avg}(A)$?
equal

d) $\mathrm{avg}(\{2 + a : a \in A\})$ or $2 + \mathrm{avg}(A)$?
equal

Note: In traditional mathematics notation, for any real numbers $a < b$, the set:

$[a,b]$ denotes the closed interval $\{x : a \le x \le b\}$;

(a, b) denotes the open interval $\{x : a < x < b\}$;

$\{a, b\}$ denotes the finite set containing only a and b.

For any set $\mathcal{S}$ and integer $n \in \mathbb{N}$, the symbol "$\mathcal{S}^n$" denotes the set of ordered n-tuples $(s_1, s_2, ..., s_n)$ with each $s_i \in \mathcal{S}$— for example, $\mathbb{R}^3$ is the set of three-tuples (x_1, x_2, x_3) of real numbers $x_i \in \mathbb{R} = (-\infty, \infty)$, a suitable set for indexing points in space, while $[0, 1]^6$ is the unit cube in six-dimensional space.