MTH135/STA104: Probability

Homework # 2 Due: Tuesday, Sep 13, 2005

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Reminder: On this and every homework, give (non-integer) numberical answers as fractions *in lowest terms* or as decimals to four significant digits—so $\pi/1000 = 0.003142$, $24/6 = 4$.

1. Let U_1 and U_2 be independent $\mathsf{Un}[0,1]$ random variables; recall that $U \sim \mathsf{Un}[0,1]$ means that, for any real number $x \in \mathbb{R}$,

$$\mathsf{P}[U \le x] = \begin{cases} 0 & -\infty < x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x < \infty, \end{cases}$$

and that, since $U_1 \perp\!\!\!\perp U_2$, we may think of (U_1, U_2) as a point drawn uniformly from the unit square. Evaluate the probabilities (art helps):

a) $\mathsf{P}[U_1 \le 0.8]$

b) $\mathsf{P}[\sqrt{U_1} \le 0.8]$

c) $\mathsf{P}[U_1 + U_2 \le 0.8]$

d) $\mathsf{P}[{U_1}^2 + U_2 \le 0.8]$

e) $\mathsf{P}[2\,U_1 - U_2 \le 0.8]$

2. Alex, Blake, and Camden like to play Scat, a three-handed card game. Alex wins twice as often as Blake, and Blake wins three times as often as poor pathetic Camden.

a) What is the chance that Alex will win the first game?

b) If they play exactly three games, what is the probability that each will win a game?

3. ($p.$ 46 prob 8) A hat contains a lot of cards, with 30% white on both sides, 50% black on one side and white on the other, and 20% black on both sides. The cards are mixed up and then a single card is drawn at random and placed on the table. If the top is black, what is the probability that the other side is white?

4. ($p.$ 46 prob 12) Give a formula for $\mathsf{P}[F \mid G^c]$ in terms of $\mathsf{P}[F]$, $\mathsf{P}[G]$ and $\mathsf{P}[F\,G]$.

5. A German Doppelkoff deck has 48 cards, 12 each of the four suits $\heartsuit$, $\spadesuit$, $\diamondsuit$, and $\clubsuit$. The deck is shuffled well and five cards are dealt, without replacement. Find the probabilities that:

a) All five cards are $\heartsuit$'s.

b) At least one card is a $\heartsuit$.

c) Exactly three of the five cards are $\heartsuit$'s.

6. A dodecahedral (twelve-sided) die is equally likely to show any of its twelve faces, numbered $\{1, 2, ..., 12\}$. If such a die is tossed ten times in a row, what is the probability that two (or more) *consecutive* tosses will both show 12, somewhere among the ten tosses?

Two solutions:

- Let $f(n)$ be the probability of at least one consecutive pair of successes in n independent tries, each with probability p of success. Then $f(0) = 0$ and $f(1) = 0$, while for $k \geq 0$,

$$f(k+2) = p^2 + q\, f(k+1) + pq\, f(k) \qquad (q \equiv 1 - p). \tag{1}$$

A direct solution is to use this relation to compute successively $f(2)$, $f(3)$,...; this can be automated a bit with a spreadsheet or a loop in any programming language.

Eqn. (1) can also be solved in closed form. The constant $f(k) \equiv 1$ satisfies Eqn. (1) (but not the boundary conditions), as does $f(k) = 1 + a\, r^k$ for any number r satisfying the homogeneous equation

$$\begin{aligned} r^{k+2} &= q\, r^{k+1} + pq\, r^k \\ r^2 &= q\, r + pq \\ 0 &= r^2 - q\, r - pq \\ r &= \frac{q \pm \delta}{2}, \quad \delta \equiv \sqrt{q^2 + 4\, pq} \end{aligned}$$

If we denote the two solutions by r_+ and r_-, then there are unique numbers a_+, a_- for which

$$f(k) \equiv 1 + a_+(r_+)^k + a_-(r_-)^k \tag{2}$$

satisfies the two boundary conditions,

$$\begin{aligned}
0 = f(0) &= 1 + a_+ + a_- \\
0 = f(1) &= 1 + a_+ r_+ + a_- r_- \\
&= 1 + (a_+ + a_-)(q/2) + (a_+ - a_-)(\delta/2) \\
&= 1 - q/2 + (a_+ - a_-)\delta/2; \text{ thus} \\
a_+ + a_- &= -1 \\
a_+ - a_- &= (q-2)/\delta \\
a_+ &= (q-2)/2\delta - 1/2 \\
a_- &= (2-q)/2\delta - 1/2
\end{aligned}$$

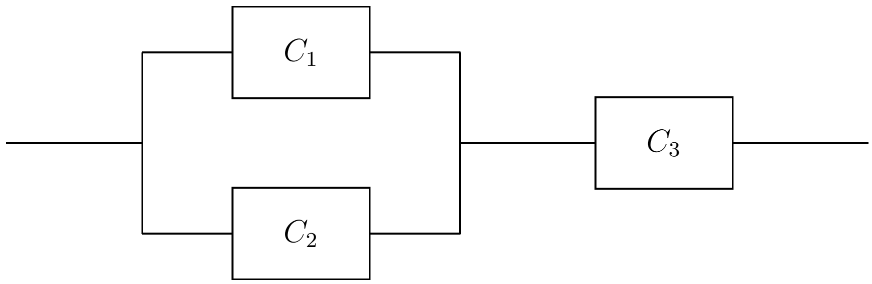
To solve the problem, set $p = 1/12$, $q = 11/12$, and $n = 10$; the answer from Eqn. (2) with the indicated values for $a_\pm$ is about 0.05700001.

- Another solution: For each integer $0 \le k \le n/2$, the number of ways to pick k numbers in the range $1, 2, ..., n$ with *no two consecutive* is the same as the way to choose $r = k+1$ integers $x_j \ge 1$ with sum $\sum x_j = S = n + 2 - k$ (why?); it's easy to show that this is $\binom{S-1}{r-1}$ in general, or $\binom{n+1-k}{k}$ for us. Thus the probability in question is

$$\begin{aligned}
1 \quad &- \quad \sum_{k=0}^{5} \binom{11-k}{k} \left(\frac{1}{12}\right)^k \left(\frac{11}{12}\right)^{10-k} \\
= 1 \quad &- \quad 1\,\frac{11^{10}}{12^{10}} - 10\,\frac{11^9}{12^{10}} - 36\,\frac{11^8}{12^{10}} - 56\,\frac{11^7}{12^{10}} - 35\,\frac{11^6}{12^{10}} - 6\,\frac{11^5}{12^{10}} \\
= 1 \quad &- \quad 0.4189 - 0.3808 - 0.1246 - 0.0176 - 0.0010 - 0.0000156,
\end{aligned}$$

or (again) about 0.05700001.

7. In the following diagram, the system breaks down if either component C_3 fails or if both components C_1 and C_2 fail (or both).



The failure probabilities the three components are $\mathsf{P}[F_1] = 0.10$, $\mathsf{P}[F_2] = 0.70$, and $\mathsf{P}[F_3] = 0.30$, respectively; all components work or fail independently. What is the probability that the system works?

8. (almost *p.* 54 prob 5) The fraction of persons in a population who have a certain disease is 0.02. A diagnostic test is available to test for the disease, but it is not perfect: a healthy person has probability 0.05 that the diagnostic test will (incorrectly) indicate presence of the disease, while a diseased person might have the disease go undetected by this diagnostic with probability 0.10. For a person selected at random from the population, find

a) The probability of a positive test result (indicating disease);

b) The probability the person selected at random *does* have the disease but that the diagnostic *does not* identify it;

c) The probability that the person is correctly diagnosed and healthy;

d) The conditional probability of disease, given that the diagnostic test result is positive (for the disease).

9. Three fair coins are tossed at once— a nickle, a dime, and a quarter.

a) What is the probability that at least one coin shows Heads?

b) What is the probability that at least \$0.32 worth of coins show heads?

c) Skyler wants to know the probability that all three coins show the same face (*i.e.*, all Heads or all Tails). She reasons that at least two of the coins *must* show the same face (after all, there are only two possibilities), and there is a 50% chance the third face will match. Is she right? Why?

d) Find the probability that all three faces show Heads, given that at least one of them does.

e) Can you find an event E with probability exactly $\mathsf{P}[E] = 1/3$? Why?

10. Brite Lites has three lightbulb factories, one each in Athens, Brevard, and Coallawalla. The probabilities of defective lightbulbs at these three factories are $\mathsf{P}[A] = 0.07$, $\mathsf{P}[B] = 0.05$, $\mathsf{P}[C] = 0.01$. A shipment contais 10 bulbs from facility A, 20 bulbs from B, and 70 bulbs from C.

a) A single light bulb is chosen at random from this shipment. If it is defective, what is the probability it came from factory A?

b) What is the probability that this entire shipment contains at least two defective bulbs?