MTH135/STA104: Probability

Homework # 2 Solution   Due: Tuesday, Sep 13, 2005

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Reminder: On this and every homework, give (non-integer) numerical answers as fractions *in lowest terms* or as decimals to four significant digits—so $\pi/1000 = 0.003142$, $24/6 = 4$.

1. Let U_1 and U_2 be independent $\mathsf{Un}[0,1]$ random variables; recall that $U \sim \mathsf{Un}[0,1]$ means that, for any real number $x \in \mathbb{R}$,

$$\mathsf{P}[U \le x] = \begin{cases} 0 & -\infty < x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x < \infty, \end{cases}$$

and that, since $U_1 \perp\!\!\!\perp U_2$, we may think of (U_1, U_2) as a point drawn uniformly from the unit square. Evaluate the probabilities (art helps):

a) $\mathsf{P}[U_1 \le 0.8] = \frac{0.8*1}{1} = 0.8$

b) $\mathsf{P}[\sqrt{U_1} \le 0.8] = \mathsf{P}[U_1 \le 0.64] = \frac{0.64*1}{1} = 0.64$

c) $\mathsf{P}[U_1 + U_2 \le 0.8] = \frac{(0.8*0.8)/2}{1} = 0.32$

d) $\mathsf{P}[{U_1}^2 + U_2 \le 0.8] = \int_0^{\sqrt{0.8}} (0.8 - U_1^2) dU = \left[0.8U_1 - \frac{U_1^3}{3}\right]_0^{\sqrt{0.8}} = 0.47703$

e) $\mathsf{P}[2\,U_1 - U_2 \le 0.8] = (0.4 * 1) + (.5 * 1)/2 = 0.65$

2. Alex, Blake, and Camden like to play Scat, a three-handed card game. Alex wins twice as often as Blake, and Blake wins three times as often as poor pathetic Camden.
So then, $A = 2B = 6C$ and $B = 3C$

a) What is the chance that Alex will win the first game?
Make a system of equations. Since, $\mathsf{P}[B] = 3\mathsf{P}[C]$ and $\mathsf{P}[A] + \mathsf{P}[B] + \mathsf{P}[C] = 1$,

then $\mathsf{P}[C] + 3\mathsf{P}[C] + 6\mathsf{P}[C] = 1$.
$\mathsf{P}[A]$wins one game is $= \frac{6}{10}$ or $\frac{3}{5}$

b) If they play exactly three games, what is the probability that each will win a game?

$\mathsf{P}[\text{each one wins a game}] = 3!\left(\frac{1}{10}\right)\left(\frac{3}{10}\right)\left(\frac{6}{10}\right) = \frac{27}{250}$

3. (*p.* 46 prob 8) A hat contains a lot of cards, with 30% white on both sides, 50% black on one side and white on the other, and 20% black on both sides. The cards are mixed up and then a single card is drawn at random and placed on the table. If the top is black, what is the probability that the other side is white?

$\mathsf{P}[WW] = 0.3$, $\mathsf{P}[BW] = 0.5$, and $\mathsf{P}[BB] = 0.2$

$$\begin{aligned}
\mathsf{P}[\text{one side is white}|\text{one side is black}] &= \frac{P[\text{one side is white and one side is black}]}{P[\text{one side is black}]} \\
&= \frac{P[\text{one side is white and one side is black}]}{P[\text{exactly one side is black or both sides are black}]} \\
&= \frac{5}{9}
\end{aligned}$$

4. (*p.* 46 prob 12) Give a formula for $\mathsf{P}[F \mid G^c]$ in terms of $\mathsf{P}[F]$, $\mathsf{P}[G]$ and $\mathsf{P}[F\,G]$.

$$\mathsf{P}[F \mid G^c] = \frac{\mathsf{P}[F\,G^c]}{\mathsf{P}[G^c]} = \frac{\mathsf{P}[F] - \mathsf{P}[F\,G]}{1 - \mathsf{P}[G]}$$

5. A German Doppelkoff deck has 48 cards, 12 each of the four suits $\heartsuit$, $\spadesuit$, $\diamondsuit$, and $\clubsuit$. The deck is shuffled well and five cards are dealt, without replacement. Find the probabilities that:

a) All five cards are $\heartsuit$'s.

$$\mathsf{P}[\text{All five cards are } \heartsuit\text{'s.}] = (\frac{12}{48})(\frac{11}{47})(\frac{10}{46})(\frac{9}{45})(\frac{8}{44}) = \frac{1}{2162}$$

b) At least one card is a $\heartsuit$.

$$\begin{aligned}
\mathsf{P}[\text{At least one card is a } \heartsuit\text{'s.}] &= 1 - \mathsf{P}[\text{no card is a } \heartsuit\ ] \\
&= 1 - \left[(\frac{36}{48})(\frac{35}{47})(\frac{34}{46})(\frac{33}{45})(\frac{32}{44})\right] \\
&= 1 - 0.2202 = 0.7798
\end{aligned}$$

c) Exactly three of the five cards are $\heartsuit$'s.

$$\mathsf{P}[\text{Exactly three of the five cards are } \heartsuit\text{'s}] = \binom{5}{2}(\frac{12}{48})(\frac{11}{47})(\frac{10}{46})(\frac{36}{45})(\frac{35}{44}) = 0.008094$$

6. A dodecahedral (twelve-sided) die is equally likely to show any of its twelve faces, numbered $\{1, 2, ..., 12\}$. If such a die is tossed ten times in a row, what is the probability that two (or more) *consecutive* tosses will both show 12, somewhere among the ten tosses?

- Let $f(n)$ be the probability of at least one consecutive pair of successes in n independent tries, each with probability p of success. Then $f(0) = 0$ and $f(1) = 0$, while for $k \geq 0$,

$$f(k+2) = p^2 + q\, f(k+1) + pq\, f(k) \qquad (q \equiv 1 - p). \tag{1}$$

A direct solution is to use this relation to compute successively $f(2)$, $f(3)$,...; this can be automated a bit with a spreadsheet or a loop in any programming language.
Eqn. (1) can also be solved in closed form. The constant $f(k) \equiv 1$ satisfies Eqn. (1) (but not the boundary conditions), as does $f(k) = 1 + a\, r^k$ for any number r satisfying the homogeneous equation

$$\begin{aligned} r^{k+2} &= q\, r^{k+1} + pq\, r^k \\ r^2 &= q\, r + pq \\ 0 &= r^2 - q\, r - pq \\ r &= \frac{q \pm \delta}{2}, \quad \delta \equiv \sqrt{q^2 + 4\, pq} \end{aligned}$$

If we denote the two solutions by r_+ and r_-, then there are unique numbers a_+, a_- for which

$$f(k) \equiv 1 + a_+(r_+)^k + a_-(r_-)^k \tag{2}$$

satisfies the two boundary conditions,

$$
\begin{aligned}
0 = f(0) &= 1 + a_+ + a_- \\
0 = f(1) &= 1 + a_+ r_+ + a_- r_- \\
&= 1 + (a_+ + a_-)(q/2) + (a_+ - a_-)(\delta/2) \\
&= 1 - q/2 + (a_+ - a_-)\delta/2; \text{ thus} \\
a_+ + a_- &= -1 \\
a_+ - a_- &= (q-2)/\delta \\
a_+ &= (q-2)/2\delta - 1/2 \\
a_- &= (2-q)/2\delta - 1/2
\end{aligned}
$$

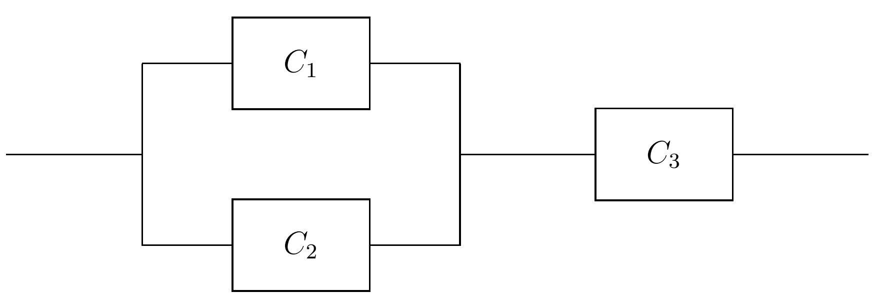
To solve the problem, set $p = 1/12$, $q = 11/12$, and $n = 10$; the answer from Eqn. (2) with the indicated values for $a_\pm$ is about 0.05700001.

- Another solution: For each integer $0 \le k \le n/2$, the number of ways to pick k numbers in the range $1, 2, ..., n$ with *no two consecutive* is the same as the way to choose $r = k+1$ integers $x_j \ge 1$ with sum $\sum x_j = S = n + 2 - k$ (why?); it's easy to show that this is $\binom{S-1}{r-1}$ in general, or $\binom{n+1-k}{k}$ for us. Thus the probability in question is

$$
\begin{aligned}
1 &- \sum_{k=0}^{5} \binom{11-k}{k} \left(\frac{1}{12}\right)^k \left(\frac{11}{12}\right)^{10-k} \\
= 1 &- 1\,\frac{11^{10}}{12^{10}} - 10\,\frac{11^9}{12^{10}} - 36\,\frac{11^8}{12^{10}} - 56\,\frac{11^7}{12^{10}} - 35\,\frac{11^6}{12^{10}} - 6\,\frac{11^5}{12^{10}} \\
= 1 &- 0.4189 - 0.3808 - 0.1246 - 0.0176 - 0.0010 - 0.0000156,
\end{aligned}
$$

or (again) about 0.05700001.

7. In the following diagram, the system breaks down if either component C_3 fails or if both components C_1 and C_2 fail (or both).



The failure probabilities the three components are $\mathsf{P}[F_1] = 0.10$, $\mathsf{P}[F_2] = 0.70$, and $\mathsf{P}[F_3] = 0.30$, respectively; all components work or fail independently. What is the probability that the system works?

$$\mathsf{P}[\text{system works}] = (0.9)(0.3)(0.7) + (0.1)(0.3)(0.7) + (0.9)(0.7)(0.7) = 0.651$$

OR

$$\mathsf{P}[\text{system works}] = (1 - 0.10 * 0.70)(1 - .3) = 0.93 * 0.7 = 0.65$$

8. (almost $p.$ 54 prob 5) The fraction of persons in a population who have a certain disease is 0.02. A diagnostic test is available to test for the disease, but it is not perfect: a healthy person has probability 0.05 that the diagnostic test will (incorrectly) indicate presence of the disease, while a diseased person might have the disease go undetected by this diagnostic with probability 0.10. For a person selected at random from the population, find

a) The probability of a positive test result (indicating disease);

$$\mathsf{P}[\text{positive test result}] = (0.05)(0.98) + (0.90)(0.02) = 0.067$$

b) The probability the person selected at random *does* have the disease but that the diagnostic *does not* identify it;

$$(0.1)(0.02) = 0.002$$

c) The probability that the person is correctly diagnosed and healthy;

$$(1 - 0.05)(1 - 0.02) = 0.931$$

d) The conditional probability of disease, given that the diagnostic test result is positive (for the disease).

$$\mathsf{P}[\text{disease} \mid \text{indicated disease}] = \frac{\mathsf{P}[\text{indicated disease} \mid \text{disease}]\mathsf{P}[\text{disease}]}{\mathsf{P}[\text{indicated disease}]} =$$

$$= \frac{\mathsf{P}[\text{indicated disease} \mid \text{disease}]\mathsf{P}[\text{disease}]}{\mathsf{P}[\text{indicated disease} \mid \text{disease}]\mathsf{P}[\text{disease}] + \mathsf{P}[\text{indicated disease } |\text{no disease}]\mathsf{P}[\text{no disease}]} =$$

$$= \frac{(0.9)(0.02)}{(0.9)(0.02) + (0.05)(0.98)} = \frac{0.18}{0.67} = 0.2687$$

9. Three fair coins are tossed at once— a nickle, a dime, and a quarter.

a) What is the probability that at least one coin shows Heads?

$$\mathsf{P}[\text{at least one coin shows Heads}] =$$
$$1 - \mathsf{P}[\text{no coin shows Heads}] = 1 - \frac{1}{8} = \frac{7}{8}$$

b) What is the probability that at least \$0.32 worth of coins show heads?

$$\mathsf{P}[\text{at least \$0.32 worth of coins show heads?}] = 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

c) Skyler wants to know the probability that all three coins show the same face (*i.e.*, all Heads or all Tails). She reasons that at least two of the coins *must* show the same face (after all, there are only two possibilities), and there is a 50% chance the third face will match. Is she right? Why?

$$\text{No, the probability is } \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

d) Find the probability that all three faces show Heads, given that at least one of them does.

$$\frac{(1/8)}{(1-1/8)} = \frac{1}{7}$$

e) Can you find an event E with probability exactly $\mathsf{P}[E] = 1/3$? Why? No; every event in this space has probability $\frac{j}{8}$ for some integer $0 \le j \le 8$; $\frac{1}{3}$ is not of that form.

10. Brite Lites has three lightbulb factories, one each in Athens, Brevard, and Coallawalla. The probabilities of defective lightbulbs at these three factories are $\mathsf{P}[A] = 0.07$, $\mathsf{P}[B] = 0.05$, $\mathsf{P}[C] = 0.01$. A shipment contains 10 bulbs from facility A, 20 bulbs from B, and 70 bulbs from C.

a) A single light bulb is chosen at random from this shipment. If it is defective, what is the probability it came from factory A?

$$\begin{aligned}
\mathsf{P}[A|\text{ Defective}] &= \frac{\mathsf{P}[\text{Defective }|A]\mathsf{P}[A]}{\mathsf{P}[\text{Defective}]} = \\
&= \frac{\mathsf{P}[\text{Defective }|A]\mathsf{P}[A]}{\mathsf{P}[\text{Defective }|A]\mathsf{P}[A] + \mathsf{P}[\text{Defective }|B]\mathsf{P}[B] + \mathsf{P}[\text{Defective }|C]\mathsf{P}[C]} = \\
&= \frac{0.07 * 0.10}{(0.07 * 0.10) + (0.05 * 0.20) + (0.70 * 0.01)} = \frac{0.007}{0.007 + 0.01 + 0.007} = \\
&= \frac{0.007}{0.024} = 0.29167
\end{aligned}$$

b) What is the probability that this entire shipment contains at least two defective bulbs?

$$\begin{aligned}
\mathsf{P}[\text{at leas two defective bulbs }] &= 1 - \mathsf{P}[\text{none or one defective bulbs}] \\
\mathsf{P}[\text{none defective bulbs }] &= 0.93^{10} * 0.95^{20} * 0.99^{70} = 0.08585 \\
\mathsf{P}[\text{one defective bulb }] &= \binom{10}{1} 0.93^{9} * 0.07 * 0.95^{20} * 0.99^{70} + \\
&+ \binom{10}{1} 0.93^{10} * 0.95^{19} * 0.05 * 0.99^{70} + \\
&+ \binom{10}{1} 0.93^{10} * 0.95^{20} * 0.99^{69} * 0.01 = \\
&= 0.1185 \\
\mathsf{P}[\text{at least two defective bulbs }] &= 1 - (0.08585 + 0.1185) = 0.79565
\end{aligned}$$