MTH135/STA104: Probability

Homework # 4 Due: Tuesday, Sep 27, 2005

Prof. Robert Wolpert

1. (from Prob 2 $p.\,121$) Find Poisson approximations to the proabilities of the following events in 1000 independent trials with probability 0.005 of success on each trial:

The mean is $\mu = 1000 \times 0.005 = 5.0$, so

a) 1 success;

$\frac{5^1}{1!}e^{-5} = 5\,e^{-5} \approx 0.03368973$

b) 2 or fewer successes;

$\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right] e^{-5} = (1 + 5 + 25/2)\,e^{-5} = 18.5\,e^{-5} \approx 0.1246520$

c) more than 3 successes.

$1 - \left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!}\right] e^{-5} = 1 - (1 + 5 + 25/2 + 125/6)\,e^{-5} \approx 0.7349741$

2. (Prob 6 $p.\,122$) A box contains 1000 balls, of which 2 are black and the rest white.

a) Which of the following is more likely to happen in 1000 draws with replacement from the box?

Fewer than 2 black balls   Exactly 2 black balls   More than 2 black balls

The Poisson approximations with $\mu = 2$ give $\left[\frac{2^0}{0!} + \frac{2^1}{1!}\right] e^{-2} = 3\,e^{-2} \approx 0.4060$, $\left[\frac{2^2}{2!}\right] e^{-2} = 2\,e^{-2} \approx 0.2707$, and $1 - \left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right] e^{-2} = 1 - 5\,e^{-2} \approx 0.3233$, respectively, so *Fewer than 2 black balls* is most likely.

b) If two series of 1000 draws are made at random from this box, what approximately is the chance they produce the same number of black balls?

We must sum over the different possibilities k for the (common) number of black balls; the chance that *both* series yield the specific number k is just the square of the individual probabilities, so the answer is

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{5^{2k}\, e^{-10}}{k!^2} &= 0.00005 + 0.00113 + 0.00709 + 0.01970 + 0.03079 \\ &\quad + 0.03079 + 0.02138 + 0.01091 + 0.00426 + 0.00132 \\ &\quad + 0.00033 + 0.00007 + 0.00001 \approx 0.12783\end{aligned}$$

3. (from Prob 9, $p.$ 122) A cereal company advertises a prize in every box of their cereal. In fact, only about 98% of the boxes have prizes. If a family buys one box of this cereal every week for a year, estimate the chance that they will collect at least 50 prizes. What assumptions are you making?

Rounding off to 52 weeks per year, the mean number of weeks *without* a prize would be $\mu = 52*0.02 = 1.04$ and the probability of failing to get a prize two or fewer times would be $\left[\frac{1.04^0}{0!} + \frac{1.04^1}{1!} + \frac{1.04^2}{2!}\right] e^{-1.04} = 2.5808\, e^{-1.04} \approx 0.9122$. We are assuming independence for the 52 trials.

4. (from Prob 1, $p.$ 127) Suppose you take a random sample of 9 tickets from a box containing 20 blue tickets and 30 white tickets.

a) What is the chance of getting exactly 4 blue tickets, if we sample *without* replacement?

$$\frac{\binom{20}{4}\binom{30}{5}}{\binom{50}{9}} = \frac{4845 \cdot 142506}{2505433700} \approx 0.2756$$

b) What is the chance of getting exactly 4 blue tickets, if we sample *with* replacement?

$$\binom{9}{4}(0.40)^4(0.60)^5 = 126 \cdot \frac{2^4 \cdot 3^5}{5^9} = \frac{489888}{1953125} \approx 0.2508$$

5. (from Prob 3, *p.* 128) A deck of 52 cards is well shuffled and dealt to four players, each receiving 13 cards. Find:

a) The probability that all the aces are in the same hand

$$4 \cdot \binom{13}{4} \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} \approx 0.01056$$

or, if you prefer,

$$4 \cdot \frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}} \approx 0.01056$$

b) Two of the Jacks in a standard deck (J♣ and J♢) have both eyes visible, while the other two (J♠ and J♡) are shown in profile revealing on a single eye. Find the conditional probability that the first player holds both one-eyed jacks, given that she holds the (one-eyed) jack of spades.

$$\frac{\mathsf{P}[\mathsf{J}\spadesuit,\ \mathsf{J}\heartsuit]}{\mathsf{P}[\mathsf{J}\spadesuit]} = \frac{\binom{2}{2}\binom{50}{11}/\binom{52}{13}}{\binom{1}{1}\binom{51}{12}/\binom{52}{13}} = \frac{\binom{50}{11}}{\binom{51}{12}} \approx 0.2353$$

c) Find the conditional probability that the first player holds both one-eyed jacks, given that she holds at least one of them

$$\frac{\mathsf{P}[\mathsf{J}\spadesuit,\ \mathsf{J}\heartsuit]}{\mathsf{P}[\mathsf{J}\spadesuit \cup \mathsf{J}\heartsuit]} = \frac{\binom{2}{2}\binom{50}{11}/\binom{52}{13}}{1 - \binom{2}{0}\binom{50}{13}/\binom{52}{13}} = \frac{\binom{50}{11}}{\binom{52}{13} - \binom{50}{13}} \approx 0.1333$$

d) Find the probability that the same player holds both one-eyed jacks *and* all the spaces.

Zero— if she holds all 13 spaces, there is no room for J♡.

6. (from Prob 5, *p.* 128) Suppose 60% of a LARGE population of voters favor candidate Z. How large a sample must be taken for there to be a 99% chance that the majority of voters in the sample will favor candidate Z?

Let's use a normal approximation to the binomial, as Pitman suggests on page 123. The interesting part here is that the sample size n isn't known at the start, but the success probability $p = 0.60$ is known. For any integer

$n \in \mathbb{N}$, the probability that a majority of the sample favors Z is

$$\begin{aligned}
\mathsf{P}[\text{majority favors } Z] &= \sum_{k>n/2} \binom{n}{k} (0.60)^k (0.40)^{n-k} \\
&\approx 1 - \Phi\left(\frac{n/2 - \mu}{\sigma}\right) \\
&= 1 - \Phi\left(\frac{n \cdot 0.50 - n \cdot 0.60}{\sqrt{n \cdot 0.60 \cdot 0.40}}\right) \\
&= 1 - \Phi\left(\frac{-0.10\sqrt{n}}{\sqrt{0.24}}\right) \\
&\approx \Phi(0.2041\sqrt{n}) \\
&\geq 0.99 \approx \Phi(2.326);
\end{aligned}$$

thus we need $n \geq (2.326/0.2041)^2 \approx 129.9$, or $n \geq 130$.

7. (from Prob 8, *p.*. 128) In a raffle with 100 tickets, five people each buy 20 tickets each. If there are 4 winning tickets in all, find the probability that:

a) The same person gets all 4 winning tickets

$$5 \cdot \frac{\binom{4}{4}\binom{96}{16}}{\binom{100}{20}} \approx 0.006178$$

b) There are four different winners

There are lots of ways to solve this; here's one. There are $\binom{5}{4} = 5$ ways to choose which four people win— let's name them A, B, C, and D. The probability that A gets one winning ticket and 19 losing tickets is $\binom{4}{1} \cdot \binom{96}{19} / \binom{100}{20}$. *IF* this happens, then the probability that B *also* gets one winning ticket and 19 losing tickets is $\binom{3}{1} \cdot \binom{77}{19} / \binom{80}{20}$, because there are 3 remaining winning tickets and 77 remaining losers. If both these happen, then the chance that C also gets one winning ticket and 19 losing tickets is $\binom{2}{1} \cdot \binom{58}{19} / \binom{60}{20}$, because 2 winning and 58 losing tickets remain; finally, if all this happens, then the chance that D gets the last winning ticket is $\binom{1}{1} \cdot \binom{39}{19} / \binom{40}{20}$. Thus the answer is

$$\binom{5}{4} \cdot \frac{\binom{4}{1}\binom{96}{19}}{\binom{100}{20}} \frac{\binom{3}{1}\binom{77}{19}}{\binom{80}{20}} \frac{\binom{2}{1}\binom{58}{19}}{\binom{60}{20}} \frac{\binom{1}{1}\binom{39}{19}}{\binom{40}{20}} \approx 0.2040$$

8. (from Prob 13, $p.\,129$) A factory which produces chips in lots of ten thousand uses the following scheme to check the quality of its product. From each lot a random sample of size 100 is taken. If the sample contains no more than two defectives, the lot is passed. If the sample contains more than two defectives, another random sample of size 100 is taken. If this sample contains no more than 2 defectives, the lot is passed. A lot that fails both tests is rejected. If a lot actually contains 500 (5%) defectives, find the chance it will pass the inspection? Approximate by sampling with replacement, and use the Poisson approximation.

Using the Poisson approximation with $\mu = 100 \cdot 5\% = 5$, the probability of passing each one of the inspections is approximately

$$p = \left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!}\right] e^{-5} = 18.5\, e^{-5} \approx 0.12465,$$

and the probability of passing both (independent, by our approximation that sampling is made with replacement) inspections is $p^2 \approx 0.01554$.

9. (from Prob 22, $p.$. 134) Suppose that, on average, 4% of the purchasers of airline tickets do not appear for the departure of their flight. Determine how many tickets should be sold for a flight on an airplane with 400 seats, so that with probability 95% everyone who appears for the departure will have a seat. What assumptions are you making?

Using a normal approximation,

$$\begin{aligned}
\mathsf{P}[\text{Everyone gets a seat}] &= \sum_{k=0}^{400} \binom{n}{k} (0.96)^k\, (0.04)^{n-k} \\
&\approx \Phi\left(\frac{400.5 - 0.96 \cdot n}{\sqrt{n \cdot 0.96 \cdot 0.04}}\right) \\
&\geq 0.95 = \Phi(1.645), \text{ so} \\
400.5 - 0.96 \cdot n &\geq 1.645\sqrt{n \cdot 0.96 \cdot 0.04} \\
&\approx 0.32235\sqrt{n},
\end{aligned}$$

a quadratic relation for $x \equiv \sqrt{n}$ with solution

$$\sqrt{n} \leq \frac{-0.3224 + \sqrt{0.3224^2 + 4 \cdot 0.96 \cdot 400.5}}{2 \cdot 0.96} = 20.258,$$

so $n \leq 20.258^2 = 410.3853$, *i.e.*, at most 410 tickets should be sold.

10. (from Prob 33, $p.\,136$)

a) How can you simulate drawing a random number equally likely to be any of the numbers $\{1, 2, 3, 4, 5\}$, using only a fair six-sided die?

One way: roll again whenever a 6 shows; the conditional probabilities of the other outcomes are all $1/5$.

b) How can you simulate flipping a fair coin, using only a *possibly biased* six-sided die whose probability of showing face i satisfies $0 < p_i < 1$ for each $i \in \{1, 2, 3, 4, 5, 6\}$ and of course $\sum p_i = 1$, but the face probabilities are otherwise unknown?

One way: roll pairs of times until one of the two rolls is even and the other odd; if both are even (or both odd) roll another pair. If the even roll precedes the odd roll, declare "Heads"; otherwise "Tails". Verify that the (conditional) probabilities are each $1/2$. Instead of "odd" and "even" you could use any complimentary events— ace and non-ace, ≤ 3 and ≥ 4, etc. The most efficient method: Roll the die twice; if both rolls show the same face, roll again until they differ. If the first of the differing faces is higher, declare "Heads"; otherwise, "Tails". Why is that more efficient?