MTH135/STA104: Probability

Homework # 5 Due: Tuesday, Oct 4, 2005

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1. Let X_1 and X_2 be the numbers on two independent rolls of a fair die; set

$$Y_1 \equiv \min(X_1, X_2) \qquad\qquad Y_2 \equiv \max(X_1, X_2)$$

a) Give the joint distribution of X_1 and X_2

b) Give the distribution of Y_1

c) Give the joint distribution of Y_1 and Y_2

d) Give the distribution of $Z \equiv Y_2 - Y_2$. What is its most likely value?

2. A fair coin is tossed three times; let X denote the number of heads on the first two tosses, Y the number of heads on the last two tosses.

a) Are X and Y independent? Prove it.

b) Give the distribution of $Z = X \cdot Y$.

3. Is it possible for two *independent* random variables X and Y (not necessarily with the same distributions) to satisfy $\mathsf{P}[X < Y] = 1$? Why?

4. For some constant $c > 0$ the random variable X takes the value $X = j$ with probability $c \cdot j$ for $j \in \{1, 2, 3, 4\}$.

a) What is the probability that X is an even number?

b) How large would you expect the *sum* to be of n independent observations X_1,...,X_n from this distribution?

c) What is the probability that n independent observations X_1,...,X_n are all equal? Simplify as much as possible.

5. (from Prob 13, p. 160) A box contains $2n$ balls of n different colors, with 2 of each color. Balls are picked at random from the box with replacement until two balls of the same color have appeared. Let X be the number of draws made.

a) Find the probability distribution for X— *i.e.*, find $\mathsf{P}[X = x]$ for every x (hint: find $\mathsf{P}[X > x]$ first, for every integer x).

b) Find the limit as $n \to \infty$ of the probability that it takes more than $\sqrt{n}$ draws to find two of the same color, if the box contains n different colors— *i.e.*, find $\lim_{n\to\infty} \mathsf{P}[X > \sqrt{n}]$ (Hint: use an exponential approximation for $\mathsf{P}[X > x]$).

6. Let X_1, X_2,..., X_n be a sequence of random variables. If each pair (X_i, X_j) are independent for $1 \le i < j \le n$, does it follow that $\{X_1, X_2, ..., X_n\}$ are indepndent? Sketch a proof or counterexample.

7. Let X be drawn uniformly from the interval $[0, 1]$ and let Y be selected uniformly from the set $\{0, 1, 2\}$, with X and Y independent. Set $Z = X + Y$. Find the function $F(z) = \mathsf{P}[Z \le z]$ for every number $-\infty < z < \infty$. (Hint: Consider the different possible values of Y separately)

8. Suppose $\mathsf{E}[X^2] = 5$, $\mathsf{E}[Y^2] = 10$, and $\mathsf{E}[X \cdot Y] = 6$.

a) Find $\mathsf{E}[(X - Y)^2]$.

b) Find the number $t \in \mathbb{R}$ that minimizes $f(t) = \mathsf{E}[(X - t\,Y)^2]$.

9. A building has 10 floors above the basement. If 12 people get onto an elevator and if each picks a floor at random from $\{1, 2, ..., 10\}$ to get out, independently of the others,

a) At how many floors do you expect the elevator to stop?

b) Let H be the highest floor the elevator reaches. Find the median m for H, *i.e.*, the number such that $\mathsf{P}[H < m] \le \frac{1}{2} \le \mathsf{P}[H \le m]$.

10. Pitman's problem 20 on p. 184 asks you to show that the distribution of any random variable X taking three values (he chooses $\{0, 1, 2\}$) is determined completely by the two "moments" $\mu_1 = \mathsf{E}[X]$ and $\mu_2 = \mathsf{E}[X^2]$, by finding formulas for $\mathsf{P}[X = x]$ in terms of μ_1 and μ_2. Show that this would *not* be true for random variables taking (at least) four different values, by finding two random variables X and Y with the *same* first two moments but *different* distributions. Suggestion: Choose $\{\pm 1, \pm 2\}$ for your points and find distinct distributions with $\mu_1 = 0$.