MTH135/STA104: Probability

Homework # 7 Due: Tuesday, Nov 1, 2005

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1. For some number $c > 0$ the random variable X has a continuous probability distribution with density function

$$f(x) = c\,x, \qquad 0 < x < 4$$

(so $f(x) = 0$ for $x \notin (0, 4)$); thus for any interval (a, b),

$$\mathsf{P}[a < X \leq b] = \int_a^b f(x)\,dx$$

Please answer the following questions about X and its density function $f(x)$:

a) Find the value of $c > 0$: $c =$ ____________________

b) Find the indicated probabilities:

- $\mathsf{P}[-2 < X \leq 2] =$ ____________________

- $\mathsf{P}[2 < X \leq 5] =$ ____________________

- $\mathsf{P}[|X - 2| > 1] =$ ____________________

2. The life-times (in years) of cathode-ray tubes (CRTs) are random variables T that satisfy

$$\mathsf{P}[T > t] = \frac{100}{t^2}, \qquad t > 10$$

for every $t > 10$. Please answer the following questions:

a) The equation above doesn't mention $t < 10$, but it does include all the information needed to determine it. Find

$$\mathsf{P}[T \le 5] = \underline{\hspace{4cm}}$$

b) Find the pdf $f(t)$ for T; be careful to give it correctly at every $t \in \mathbb{R}$.

$$f(t) = \underline{\hspace{4cm}}$$

c) Evaluate the expected value of T:

$$\mathsf{E}[T] = \underline{\hspace{4cm}}$$

d) Find the conditional probability of failure in the next year for a working CRT that is now t years old for each $t \ge 10$:

$$\mathsf{P}[T \le t+1 \mid T > t] = \underline{\hspace{4cm}}$$

Which is more likely to last another year, a 10-year old CRT or a 20-year old CRT?

e) Find the (instantaneous) hazard,

$$h(t) = \underline{\hspace{4cm}}$$

Does it increase, decrease, or stay constant?

3. In our fishing example we found that the number of fish caught in t hours, X_t, had a Poisson probability distribution with mass function

$$\mathsf{P}[X_t = x] = \frac{(\lambda t)^x}{x!} e^{-\lambda t}, \qquad x = 0, 1, 2, \cdots$$

with mean $\mu = \lambda t$, if we catch λ fish per hour on average and if the numbers of fish caught in disjoint time intervals are independent.

Find the probability density function $f_k(t)$ for T_k, the time at which the k^{th} fish is caught. Hint: Express the *event* $T_k \le t$ in terms of X_t and use the Poisson probabilities above. Half-credit for the case $k = 1$ of the first fish caught.

$$f_k(t) = \underline{\hspace{4cm}}$$

4. If X is a positive random variable with pdf $f(x)$ and CDF $F(x)$, so

$$\begin{aligned} F(x) &= \mathsf{P}[X \le x] \\ &= 0, \qquad x < 0 \\ f(x) &= F'(x) \\ &= 0, \qquad x < 0 \end{aligned}$$

and if $Y \equiv \sqrt{X}$, find the CDF $G(y) = \Pr[Y \le y]$ and pdf $g(y) = G'(y)$ in terms of $f(x)$ and $F(x)$.

5. For each question below, give an example or a brief explanation of why none is possible:

a) If X has a continuous distribution, can a function $Y = g(X)$ possibly have a discrete distribution with $\mathsf{P}[Y = y] > 0$ for some y?

b) If X has a discrete distribution with finitely-many or countably-many possible values $\{x_j\}$, can a function $Y = g(X)$ possibly have a continuous distribution with density function $g(y)$, so $\mathsf{P}[a < Y \le b] = \int_a^b g(y)\,dy$ for all $a < b$?

c) Is it possible for the density function $f(x)$ for some random variable X to satisfy $f(x) > 1$ at any point x?

d) Is it possible for a density function $f(x)$ to have a strictly positive lower bound on the positive half-line, *i.e.*, to satisfy $f(x) \ge \epsilon$ for every $x \in (0, \infty)$, for some fixed positive number $\epsilon > 0$?

e) Is it possible for the cumulative distribution $F[x] = \mathsf{P}[X \le x]$ to be strictly increasing (*i.e.*, satisfy $F(a) < F(b)$ for every $a < b$) on the entire positive half-line $0 < x < \infty$?

6. Let X be uniformly distributed on the interval $[-1, 2]$ and let $Y = X^2$. Find the probability density function $f(y)$ for Y, correct at every point $y \in \mathbb{R}$.

7. Let X be a random variable with mean $\mathsf{E}[X] = \mu$ and variance $\mathsf{Var}[X] = \mathsf{E}[(X - \mu)^2] = \sigma^2$. Define a function $\phi(a)$ by

$$\phi(a) = \mathsf{E}[(X - a)^2]$$

(this is the "expected squared error" for our best guess a of what value X might take). Find the point a^* where $\phi(a)$ takes on its minimum value $\phi(a^*)$, and find what that minimum value is. HINT: It does *not* matter whether X is continuous or discrete— you should not need to do any integrals or sums.

8. Let X be a random variable with mean $\mathsf{E}[X] = \mu$ and variance $\mathsf{Var}[X] = \mathsf{E}[(X - \mu)^2] = \sigma^2$, with a continuous distribution with density function $f(x)$. Define a function $\psi(a)$ by

$$\psi(a) = \mathsf{E}|X - a|$$

(this is the "expected absolute error" for our best guess a of what value X might take). Find the point $a^\#$ where $\psi(a)$ takes on its minimum value $\psi(a^\#)$, and find what that minimum value is. Can you find an example where the minimizing value of $\psi(a)$ is not unique?