1. For some number $c > 0$ the random variable $X$ has a continuous probability distribution with density function

$$f(x) = c x, \quad 0 < x < 4$$

(so $f(x) = 0$ for $x \notin (0, 4)$); thus for any interval $(a, b)$,

$$P[a < X \leq b] = \int_a^b f(x) \, dx$$

Please answer the following questions about $X$ and its density function $f(x)$:

a) Find the value of $c > 0$: $c = \ldots$

$$1 = P[0 < X \leq 4] = \int_0^4 c \, dx = c 4^2 / 2 = 8c,$$

so evidently $c = \frac{1}{8}$.

b) Find the indicated probabilities:

- $P[-2 < X \leq 2] = P[0 < X \leq 2] = \frac{1}{8}2^2 / 2 = \frac{1}{4}$
- $P[2 < X \leq 5] = P[2 < X \leq 4] = \frac{1}{8}(4^2 - 2^2) / 2 = \frac{3}{4}$
- $P[|X - 2| > 1] = 1 - P[1 < X \leq 3] = 1 - \frac{1}{8}(3^2 - 1^2) / 2 = \frac{1}{2}$
2. The life-times (in years) of cathode-ray tubes (CRTs) are random variables \( T \) that satisfy

\[
P[T > t] = \frac{100}{t^2}, \quad t > 10
\]

for every \( t > 10 \). Please answer the following questions:

a) The equation above doesn’t mention \( t < 10 \), but it does include all the information needed to determine it. Find

\[
P[T \leq 5] = 0, \text{ since } P[T \leq 10] = 1 - P[T > 10] = 1 - 1 = 0
\]

b) Find the pdf \( f(t) \) for \( T \); be careful to give it correctly at every \( t \in \mathbb{R} \).

\[
f(t) = \begin{cases} \frac{200}{t^3} & t > 10 \\ 0 & t \leq 10 \end{cases}
\]

c) Evaluate the expected value of \( T \):

\[
E[T] = \int_{10}^{\infty} t \frac{200}{t^3} \, dt = \int_{10}^{\infty} 200 t^{-2} \, dt = 20
\]

d) Find the conditional probability of failure in the next year for a working CRT that is now \( t \) years old for each \( t \geq 10 \):

\[
P[T \leq t + 1 \mid T > t] = 1 - \frac{P[T > t + 1]}{P[T > t]} = 1 - \frac{100/(t+1)^2}{100/t^2} = 1 - \left( \frac{t}{t+1} \right)^2
\]

Which is more likely to last another year, a 10-year old CRT or a 20-year old CRT?

The 20-year-old CRT has a better survival chance—

\[
\frac{400}{441} \approx 90.7\% > 82.6\% = \frac{100}{121}
\]

e) Find the (instantaneous) hazard,

\[
h(t) = \frac{f(t)}{1 - F(t)} = \frac{200/t^3}{100/t^2} = 2/t, \quad t > 10
\]

Does it increase, decrease, or stay constant?

It decreases for \( t > 10 \)
3. In our fishing example we found that the number of fish caught in $t$ hours, $X_t$, had a Poisson probability distribution with mass function

$$P[X_t = x] = \frac{\lambda^x e^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \ldots$$

with mean $\mu = \lambda t$, if we catch $\lambda$ fish per hour on average and if the numbers of fish caught in disjoint time intervals are independent.

Find the probability density function $f_k(t)$ for $T_k$, the time at which the $k$th fish is caught. Hint: Express the event $T_k \leq t$ in terms of $X_t$ and use the Poisson probabilities above. Half-credit for the case $k = 1$ of the first fish caught.

$$f_k(t) = \phantom{\text{Expression}}$$

First find the CDF:

$$F_1(t) = P[T_1 \leq t] = P[X_t \geq 1] = 1 - P[X_t = 0] = 1 - e^{-\lambda t} \quad \text{for } t > 0, \quad = 0 \quad \text{for } t \leq 0.$$  

Taking derivatives,

$$f_1(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

4. If $X$ is a positive random variable with pdf $f(x)$ and CDF $F(x)$, so

$$F(x) = P[X \leq x] = 0, \quad x < 0$$

$$f(x) = F'(x) = 0, \quad x < 0$$

and if $Y = \sqrt{X}$, find the CDF $G(y) = \Pr[Y \leq y]$ and pdf $g(y) = G'(y)$ in terms of $f(x)$ and $F(x)$.

$$G(y) = P[\sqrt{X} \leq y] = P[X \leq y^2], \quad y \geq 0$$

$$g(y) = 2y f(y^2), \quad y > 0$$
while \( G(y) = g(y) = 0 \) for \( y < 0 \).

5. For each question below, give an example or a brief explanation of why none is possible:

a) If \( X \) has a continuous distribution, can a function \( Y = g(X) \) possibly be a discrete distribution with \( P[Y = y] > 0 \) for some \( y \)?

Yes— for example, let \( X \) be uniformly distributed on \((0, 1)\) and let \( g(x) = 1 \) for \( x \leq \frac{1}{2} \), \( g(x) = 0 \) for \( x > \frac{1}{2} \).

b) If \( X \) has a discrete distribution with finitely-many or countably-many possible values \( \{x_j\} \), can a function \( Y = g(X) \) possibly have a continuous distribution with density function \( g(y) \), so \( P[a < Y \leq b] = \int_a^b g(y) dy \) for all \( a < b \)?

No— if \( X \) has only countably-many values, then \( Y = g(X) \) also has only countably-many values, and cannot have a density function.

c) Is it possible for the density function \( f(x) \) for some random variable \( X \) to satisfy \( f(x) > 1 \) at any point \( x \)?

Sure. Let \( X \) be uniform on the interval \((0, \frac{1}{2})\), for example; then its density function \( f(x) = 2 \) on \((0, \frac{1}{2})\) (and zero elsewhere).

d) Is it possible for a density function \( f(x) \) to have a strictly positive lower bound on the positive half-line, \( i.e., f(x) \geq \epsilon \) for every \( x \in (0, \infty) \), for some fixed positive number \( \epsilon > 0 \)?

No. This would imply that \( P[X < x] \geq \epsilon x \) for every \( x > 0 \), which would be bigger than one for \( x > 1/\epsilon \), violating the rules about probabilities.

e) Is it possible for the cumulative distribution \( F[x] = P[X \leq x] \) to be strictly increasing \( (i.e., f(x) < f(b) \) for every \( a < b \) \) on the entire positive half-line \( 0 < x < \infty \)?

Yes again— let \( X \) have density function \( e^{-x} \) for \( x > 0 \), for example, to find \( F(x) = 1 - e^{-x} \) on \((0, \infty)\), strictly increasing.

6. Let \( X \) be uniformly distributed on the interval \([-1, 2]\) and let \( Y = X^2 \). Find the probability density function \( f(y) \) for \( Y \), correct at every point \( y \in \mathbb{R} \).

Here \( Y = g(x) \) with \( f_x(x) = \frac{1}{3}, \quad -1 < x < 2, \ g(x) = x^2, \ g'(x) = 2x, \) and \( g^{-1}(y) = \pm \sqrt{y} \); thus
\[ f(y) = \sum_{x : x^2 = y} \frac{1}{2|x|/3} 1_{(-1,2)}(x) = \begin{cases} 
0 & y < 0 \\
\frac{1}{3\sqrt{y}} & 0 \leq y < 1 \\
\frac{1}{6\sqrt{y}} & 1 \leq y < 4 \\
0 & 4 \leq y. 
\end{cases} \]

The answer may also be found by first computing the CDF,

\[ F(y) = P[X^2 \leq y] = \begin{cases} 
0 & y < 0 \\
\frac{2\sqrt{y}}{3} & 0 \leq y < 1 \\
\frac{\sqrt{y}}{3} & 1 \leq y < 4 \\
1 & 4 \leq y, 
\end{cases} \]

from which the answer follows by differentiation.

7. Let \( X \) be a random variable with mean \( E[X] = \mu \) and variance \( \text{Var}[X] = E[(X - \mu)^2] = \sigma^2 \). Define a function \( \phi(a) \) by

\[ \phi(a) = E[(X - a)^2] \]

(this is the “expected squared error” for our best guess \( a \) of what value \( X \) might take). Find the point \( a^* \) where \( \phi(a) \) takes on its minimum value \( \phi(a^*) \), and find what that minimum value is. HINT: It does not matter whether \( X \) is continuous or discrete— you should not need to do any integrals or sums.

Expanding,

\[
\phi(a) = E[(X - a)^2] = E[X^2 - 2Xa + a^2] = E[X^2] - 2aE[X] + a^2
\]

Differentiating with respect to \( a \),

\[
\phi'(a) = -2E[X] + 2a = 0 \iff a = a^* = E[X],
\]

so \( a^* = \mu \) and \( \phi(a^*) = E[(X - \mu)^2] = \sigma^2 \).
8. Let $X$ be a random variable with mean $\mathbb{E}[X] = \mu$ and variance $\text{Var}[X] = \mathbb{E}[(X - \mu)^2] = \sigma^2$, with a continuous distribution with density function $f(x)$. Define a function $\psi(a)$ by

$$\psi(a) = \mathbb{E}|X - a|$$

(this is the “expected absolute error” for our best guess $a$ of what value $X$ might take). Find the point $a^\#$ where $\psi(a)$ takes on its minimum value $\psi(a^\#)$, and find what that minimum value is. Can you find an example where the minimizing value of $\psi(a)$ is not unique?

For any $a \in \mathbb{R}$,

$$\psi(a) = \mathbb{E}|X - a| = \int_{x<a} (a - x) f(x) \, dx + \int_{a<x} (x - a) f(x) \, dx$$

Differentiating with respect to $a$,

$$\psi'(a) = \int_{x<a} f(x) \, dx - \int_{a<x} f(x) \, dx = \mathbb{P}[X < a] - \mathbb{P}[X > a] = 0 \Leftrightarrow \mathbb{P}[X < a] = \frac{1}{2} = \mathbb{P}[X > a]$$

so $a^\# = m$ is a median of $X$. This is unique if $f(x) > 0$ for $x$ near $m$, but not if $f(x) = 0$ on some interval with probability $\frac{1}{2}$ on each side—for example, not if $X$ is drawn uniformly from the set $[-2, -1] \cup [1, 2]$. 

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