MTH135/STA104: Probability

Homework # 8 Due: Tuesday, Nov 8, 2005

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1. Define a function $f(x, y)$ on the plane $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} 1/x & 0 < y < x < 1 \\ 0 & \text{other } x, y \end{cases}$$

a) Show that $f(x, y)$ is a joint probability density function. What do you have to check?

If $f(x, y)$ is the joint p.d.f. for X and Y, find:

b) The marginal density functions

$f_x(x) =$ ____________ $f_y(y) =$ ____________

c) The expectations

$\mathsf{E}X =$ ____________ $\mathsf{E}Y =$ ____________

2. Let X and Y be two independent random variables, each with the uniform distribution on $(0, 1)$. Let $M = \min(X, Y)$ be the smaller of the two.

a) Represent the event $M > x$ as a region in the plane, and find the probability $\mathsf{P}[M > x]$ as the area of this region.

b) Use your result above to find the density function for M. Plot both CDF and pdf for M on the range $-1 \le x \le 2$.

3. Let $U_1, U_2, \cdots, U_n$ be n independent uniform random variables drawn from the set $(0,1)$. Order them from least to greatest and call the re-ordered variables $U_{(1)}, U_{(2)}, \cdots, U_{(n)}$; for example, $U_{(1)} \equiv \min\{U_1, U_2, \cdots, U_n\}$. Let $0 < x < y < 1$.

a) Find and justify a simple formula for $\mathsf{P}[U_{(1)} > x, U_{(n)} \le y]$.

b) Now find a simple formula for $\mathsf{P}[U_{(1)} \le x, U_{(n)} \le y]$.

c) Give the joint pdf for $U_{(1)}$ and $U_{(n)}$. Be careful about the range of x, y where your formula is correct— try to give a correct one for *all* x, y.

4. A point (x, y) is said to be drawn "uniformly" from a set $A \subset \mathbb{R}^2$ in the plane if the joint density function $f(x, y)$ has some constant value $c > 0$ for $(x, y) \in A$, and is zero outside A. Let X and Y be the coordinates of a point drawn uniformly from the triangle A with corners $(0,0)$, $(0,2)$, and $(2,0)$.

a) What is the constant value of the pdf $f(x, y)$ inside A? Why?

b) Let $R = \sqrt{X^2 + Y^2}$ be the distance of (X, Y) from the origin. Find the probability $\mathsf{P}[R \le 1]$ (Hint: draw a picture— no integration is needed)

c) Give the joint CDF $F(x, y) = \mathsf{P}[X \le x,\ Y \le y]$ correctly at *every* point $x, y \in \mathbb{R}^2$. (Hint: separate into several cases; draw pictures).

5. A straight stick is broken at random in two places chosen independently and uniformly along the length of the stick. What is the probability that the pieces can be arranged to form a triangle?

6. The random variables X and Y have joint density function

$$f(x, y) = 12xy(1 - x) \qquad 0 < x < 1, 0 < y < 1$$

and equal to zero otherwise.

a) Are X and Y independent? Why?

b) Find $\mathsf{E}X$.

c) Find $\mathsf{E}Y$.

d) Find $\mathsf{Var}X$.

e) Find $\mathsf{Var}Y$.

7. Let $X_1, X_2, \cdots, X_n$ be independent, each with the $\mathsf{Ex}(\lambda)$ distribution (so each has mean $1/\lambda$). Let $V = \min\{X_i\}$ and $W = \max\{X_i\}$ be their minimum and maximum, respectively. Find the joint density function for V and W.

8. Let T_1 and T_4 be the times of the first and fourth arrival in a Poisson process with rate λ, as in text Section 4.2 or in Prob. 3 of Homework 7. For $0 < s < t < \infty$, let X be the number of events in the period $(0, s]$ and let Y be the number of events in the time period $(s, t]$; note X and Y have independent Poisson distributions with means λs and $\lambda(t-s)$, respectively.

a) Express the *events* $[s \le T_1]$ and $[T_4 \le t]$ in terms of the random variables X and Y. You needn't compute their probabilities.

b) Express the event $[s < T_1,\ T_4 \le t]$ in terms of the random variables X and Y.

c) Express the event $[T_1 \le s,\ T_4 \le t]$ in terms of the random variables X and Y.

d) (extra credit) Find the joint pdf for T_1 and T_4.