

MTH135/STA104: Probability

Homework # 8

Due: Tuesday, Nov 8, 2005

Prof. Robert Wolpert

1. Define a function $f(x, y)$ on the plane \mathbb{R}^2 by

$$f(x, y) = \begin{cases} 1/x & 0 < y < x < 1 \\ 0 & \text{other } x, y \end{cases}$$

- a) Show that $f(x, y)$ is a joint probability density function. What do you have to check?

Two things: $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$, and

$$\iint_{\mathbb{R}^2} f(x, y) dx dy = \int_0^1 \left[\int_0^x \frac{1}{x} dy \right] dx = \int_0^1 1 dx = 1$$

If $f(x, y)$ is the joint p.d.f. for X and Y , find:

- b) The marginal density functions

$$f_x(x) = \underline{\hspace{2cm}} \quad f_y(y) = \underline{\hspace{2cm}}$$

$$f_x(x) = \int_0^x \frac{1}{x} dy = 1, \quad 0 < x < 1; \quad f_y(y) = \int_y^1 \frac{1}{x} dx = -\log y, \quad 0 < y < 1.$$

- c) The expectations

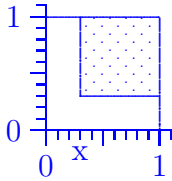
$$EX = \underline{\hspace{2cm}} \quad EY = \underline{\hspace{2cm}}$$

$EX = \int_0^1 x dx = \frac{1}{2}$; easiest way for Y is to use the joint pdf,

$$EY = \iint_{0 < y < x < 1} y \frac{1}{x} dy dx = \int_0^1 \frac{x^2}{2x} dx = \frac{1}{4}.$$

2. Let X and Y be two independent random variables, each with the uniform distribution on $(0, 1)$. Let $M = \min(X, Y)$ be the smaller of the two.

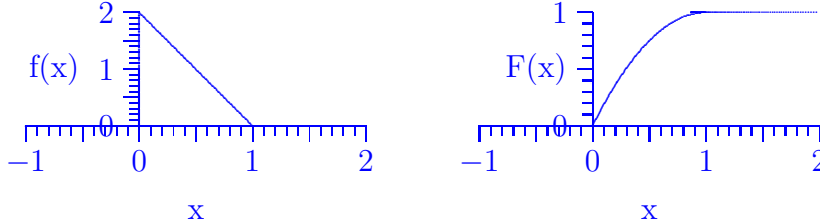
a) Represent the event $M > x$ as a region in the plane, and find the probability $P[M > x]$ as the area of this region.



Evidently $P[M > x] = (1 - x)^2$ for $0 < x < 1$.

b) Use your result above to find the density function for M . Plot both CDF and pdf for M on the range $-1 \leq x \leq 2$.

The CDF is $F(x) = 1 - P[M > x] = 1 - (1 - x)^2$ for $0 < x < 1$, giving pdf $f(x) = 2(1 - x)$ for $0 < x < 1$:



3. Let U_1, U_2, \dots, U_n be n independent uniform random variables drawn from the set $(0, 1)$. Order them from least to greatest and call the re-ordered variables $U_{(1)}, U_{(2)}, \dots, U_{(n)}$; for example, $U_{(1)} \equiv \min\{U_1, U_2, \dots, U_n\}$. Let $0 < x < y < 1$.

a) Find and justify a simple formula for $P[U_{(1)} > x, U_{(n)} \leq y]$.

The indicated event occurs if and only if *all* the U_j 's fall within the interval $(x, y]$, which happens with probability

$$P[U_{(1)} > x, U_{(n)} \leq y] = P\left\{\bigcap_{j=1}^n [x < U_j \leq y]\right\} = (y - x)^n$$

b) Now find a simple formula for $P[U_{(1)} \leq x, U_{(n)} \leq y]$.

$$\begin{aligned} P[U_{(1)} \leq x, U_{(n)} \leq y] &= P[U_{(1)} > 0, U_{(n)} \leq y] - P[U_{(1)} > x, U_{(n)} \leq y] \\ &= y^n - (y - x)^n, \quad 0 < x < y < 1. \end{aligned}$$

c) Give the joint pdf for $U_{(1)}$ and $U_{(n)}$. Be careful about the range of x, y where your formula is correct— try to give a correct one for *all* x, y .

Taking derivatives w.r.t. x and y gives

$$f(x, y) = n(n-1)(y-x)^{n-2}, \quad 0 < x < y < 1$$

with $f(x, y) = 0$ for other x, y .

4. A point (x, y) is said to be drawn “uniformly” from a set $A \subset \mathbb{R}^2$ in the plane if the joint density function $f(x, y)$ has some constant value $c > 0$ for $(x, y) \in A$, and is zero outside A . Let X and Y be the coordinates of a point drawn uniformly from the triangle A with corners $(0, 0)$, $(0, 2)$, and $(2, 0)$.

a) What is the constant value of the pdf $f(x, y)$ inside A ? Why?

One over the area of A , or $\frac{1}{2}$, since the integral of $f(x, y)$ over A must be one.

b) Let $R = \sqrt{X^2 + Y^2}$ be the distance of (X, Y) from the origin. Find the probability $P[R \leq 1]$ (Hint: draw a picture— no integration is needed)

That quarter-circle, with area $\pi/4$, lies completely within the triangle A , so the probability is simply $\frac{1}{2} \times (\pi/4) = \pi/8$.

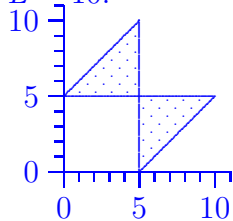
c) Give the joint CDF $F(x, y) = P[X \leq x, Y \leq y]$ correctly at *every* point $x, y \in \mathbb{R}^2$. (Hint: separate into several cases; draw pictures).

$$F(x, y) = \begin{cases} 0 & x \leq 0 \text{ or } y \leq 0 \\ xy/2 & 0 < x, 0 < y, x+y \leq 2 \\ x+y-1-(x^2+y^2)/4 & 0 < x \leq 2, 0 < y \leq 2, x+y > 2 \\ x-x^2/4 & 0 < x \leq 2, y > 2 \\ y-y^2/4 & x > 2, 0 < y \leq 2 \\ 1 & x > 2, y > 2 \end{cases}$$

5. A straight stick is broken at random in two places chosen independently and uniformly along the length of the stick. What is the probability that the pieces can be arranged to form a triangle?

Denote by L the length of the stick, and by x and y the two break points. The pieces will form a triangle if none is longer than the sum of the others; if $0 < x < y < L$ the conditions are that $x < (L-x)$ (or $x < L/2$), $(L-y) < y$

(or $y > L/2$), and $(y - x) < x + (L - y)$ (or $y < x + L/2$). Thus the part of the allowable region with $x < y$ is bounded by the lines $x = L/2$, $y = L/2$, and $y = x + L/2$; here is a diagram with the allowable region shaded, for $L = 10$:



Evidently each half of the shaded region is a triangle with area $\frac{1}{2}(L/2)^2$, so the total area is $L^2/4$ and the indicated probability is

$$P[\text{Pieces form a triangle}] = 1/4.$$

6. The random variables X and Y have joint density function

$$f(x, y) = 12xy(1 - x) \quad 0 < x < 1, 0 < y < 1$$

and equal to zero otherwise.

a) Are X and Y independent? Why?

Yes, the joint pdf is a product of $f_y(y) = 2y$, $0 < y < 1$ and $f_x(x) = 6x(1 - x)$, $0 < x < 1$.

b) Find EX .

$$\int_0^1 x 6x(1 - x) dx = \int (6x^2 - 6x^3) dx = \left(2x^3 - \frac{3}{2}x^4\right) \Big|_{x=0}^{x=1} = \frac{1}{2}.$$

c) Find EY .

$$\int_0^1 y 2y dy = \left(2y^3/3\right) \Big|_{y=0}^{y=1} = \frac{2}{3}.$$

d) Find $\text{Var}X$.

$$\text{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \int_0^1 x^2 6x(1-x) dx - \frac{1}{4} = \left(\frac{6x^4}{4} - \frac{6x^5}{5}\right)\Big|_{x=0}^{x=1} - \frac{1}{4} = \frac{1}{20}.$$

e) Find $\text{Var}Y$.

$$\text{Var}Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \int_0^1 y^2 2y dy - \frac{4}{9} = \left(\frac{2y^4}{4}\right)\Big|_{y=0}^{y=1} - \frac{4}{9} = \frac{1}{18}.$$

7. Let X_1, X_2, \dots, X_n be independent, each with the $\text{Ex}(\lambda)$ distribution (so each has mean $1/\lambda$). Let $V = \min\{X_i\}$ and $W = \max\{X_i\}$ be their minimum and maximum, respectively. Find the joint density function for V and W .

For $0 \leq v \leq w < \infty$ we have $\mathbb{P}[v < X_i \leq w] = e^{-\lambda v} - e^{-\lambda w}$, so

$$\begin{aligned} \mathbb{P}[V > v, \quad W \leq w] &= \mathbb{P}\left[\cap_{i=1}^n [v < X_i \leq w]\right] \\ &= \prod_{i=1}^n [e^{-\lambda v} - e^{-\lambda w}] \\ &= [e^{-\lambda v} - e^{-\lambda w}]^n, \quad \text{so} \\ F(v, w) &= \mathbb{P}[V \leq v, \quad W \leq w] \\ &= \mathbb{P}[V > 0, \quad W \leq w] - \mathbb{P}[V > v, \quad W \leq w] \\ &= [1 - e^{-\lambda w}]^n - [e^{-\lambda v} - e^{-\lambda w}]^n \end{aligned}$$

$$f(v, w) = \frac{\partial^2}{\partial v \partial w} F(v, w) = n(n-1)\lambda^2 [e^{-\lambda v} - e^{-\lambda w}]^{n-2} e^{-\lambda(v+w)}, \quad 0 \leq v \leq w < \infty;$$

$f(v, w) = 0$ for u, v outside that range.

8. Let T_1 and T_4 be the times of the first and fourth arrival in a Poisson process with rate λ , as in text Section 4.2 or in Prob. 3 of Homework 7. For $0 < s < t < \infty$, let X be the number of events in the period $(0, s]$ and let Y be the number of events in the time period $(s, t]$; note X and Y have independent Poisson distributions with means λs and $\lambda(t-s)$, respectively.

a) Express the *events* $[s < T_1]$ and $[T_4 \leq t]$ in terms of the random variables X and Y . You needn't compute their probabilities.

$[s < T_1] = [X = 0]$; the first arrival is later than s if and only if no arrivals have come by time s . The events $[T_4 \leq t] = [X + Y \geq 4]$ are identical, asserting that the fourth event occurred before time t or, equivalently, that the total number of events $X + Y$ during the time interval $(0, t] = (0, s] \cup (s, t]$ is at least four.

b) Express the event $[s < T_1, T_4 \leq t]$ in terms of the random variables X and Y .

$$[s < T_1, T_4 \leq t] = [X = 0, Y \geq 4].$$

c) Express the event $[T_1 \leq s, T_4 \leq t]$ in terms of the random variables X and Y .

$$\begin{aligned} [T_1 \leq s, T_4 \leq t] &= [T_1 > 0, T_4 \leq t] \cap [T_1 > s, T_4 \leq t]^c \\ &= [X + Y \geq 4] \cap [X = 0, Y \geq 4]^c \\ &= [X > 0, X + Y \geq 4] \end{aligned}$$

d) (extra credit) Find the joint pdf for T_1 and T_4 .

Fix any numbers $0 \leq s \leq t < \infty$ and let X be the number of events in the interval $(0, s]$ and let Y be the number of events in the interval $(s, t]$; these have independent Poisson distributions with means λs and $\lambda(t - s)$, respectively and their sum $Z = X + Y$, the number of events in the interval $(0, t]$, has a Poisson distribution with mean λt . Now

$$\begin{aligned} P[T_1 \leq s, T_4 \leq t] &= P[T_4 \leq t] - P[T_1 > s, T_4 \leq t] \\ &= P[Z \geq 4] - P[X = 0, Y \geq 4] \\ &= \sum_{k=4}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} - e^{-\lambda s} \sum_{k=4}^{\infty} e^{-\lambda(t-s)} \frac{\lambda^k (t-s)^k}{k!} \\ &= e^{-\lambda t} \sum_{k=4}^{\infty} \frac{\lambda^k [t^k - (t-s)^k]}{k!} \\ f(s, t) &= \frac{\partial}{\partial t} \left(e^{-\lambda t} \sum_{k=4}^{\infty} \frac{(t-s)^{k-1}}{(k-1)!} \lambda^k \right) \\ &= -\lambda e^{-\lambda t} \sum_{k=4}^{\infty} \frac{(t-s)^{k-1}}{(k-1)!} \lambda^k + e^{-\lambda t} \sum_{k=4}^{\infty} \frac{(t-s)^{k-2}}{(k-2)!} \lambda^k \\ &= e^{-\lambda t} \lambda^4 (t-s)^2 / 2, \quad 0 < s < t < \infty; \end{aligned}$$

the pdf is zero for s, t outside this range.