MTH135/STA104: Probability

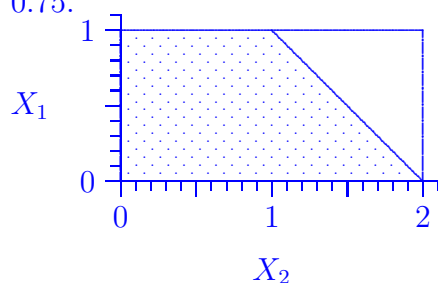Homework # 9   Due: Tuesday, Nov 15, 2005

Prof. Robert Wolpert

1. Let $X_1 \sim \mathsf{Un}(0,1)$ and $X_2 \sim \mathsf{Un}(0,2)$ be independent, and let $Y = X_1 + X_2$. Find:

   a) The probability $\mathsf{P}[Y \le 2] = $ _____

   $\mathsf{P}[Y \le 2] = 0.75$. Note that (X_1, X_2) is uniformly drawn from the rectangle below; the (shaded) set where $Y \le 2$ has area 1.5 and so probability 0.75.

   b) The density function for Y, $f(y) = $ _____

   From c) below, taking a derivative w.r.t. y,

$$f(y) = \begin{cases} \frac{y}{2} & 0 \le y < 1 \\ \frac{1}{2} & 1 \le y < 2 \\ \frac{y-3}{2} & 2 \le y < 3 \\ 0 & \text{otherwise.} \end{cases}$$

   c) The c.d.f. for Y, $F(y) = $ _____

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From a series of pictures much like that from part a), we have:

$$F(y) = \begin{cases} 0 & y < 0 \\ y^2/4 & 0 \le y < 1 \\ \frac{2y-1}{4} & 1 \le y < 2 \\ 1 - \frac{(3-y)^2}{4} & 2 \le y < 3 \\ 1 & 3 \le y \end{cases}$$

2. An auto parts customer must wait in one line to get the part she needs, then wait in another line to pay for the part to complete her purchase. The waiting time in the two lines are independent exponentially-distributed random variables with rates α and β, respectively.

a) Find the density function for the total wait time.

Denote the individual wait times by X and Y, and let $S = X + Y$. Obviously the pdf for S is $f(s) = 0$ for $s < 0$ and, if $\alpha \ne \beta$, then for $s > 0$ it is given by:

$$\begin{aligned} f(s) &= \int_0^s (\alpha e^{-\alpha x})(\beta e^{-\beta(s-x)}) \, dx \\ &= \alpha\beta e^{-\beta s} \int_0^s e^{(\beta-\alpha)x} \, dx \quad (1) \\ &= \alpha\beta e^{-\beta s} \frac{e^{(\beta-\alpha)s} - 1}{\beta - \alpha} \\ &= \frac{\alpha\beta}{\beta - \alpha} \left(e^{-\alpha s} - e^{-\beta s} \right), \quad s > 0 \end{aligned}$$

while for $\alpha = \beta$, by L'Hospital's rule or by integrating (1), the density is just

$$f(s) = \alpha^2 s e^{-\alpha s}, \quad s > 0.$$

b) Find the expected total waiting time in terms of α and β (Hint: No integration is necessary)

$$\mathsf{E}[X + Y] = \mathsf{E}X + \mathsf{E}Y = \frac{1}{\alpha} + \frac{1}{\beta}$$

c) Find the SD of the total waiting time (Same hint)

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$$\mathsf{Var}[X+Y] = \mathsf{Var}X + \mathsf{Var}Y = \frac{1}{\alpha^2} + \frac{1}{\beta^2}, \text{ so}$$

$$\mathsf{SD}[X+Y] = \sqrt{\frac{1}{\alpha^2} + \frac{1}{\beta^2}}$$

3. Let X be the number on a fair six-sided die, and let $Y \sim \mathsf{Un}(0,1)$ be drawn from the standard uniform distribution; set $Z = X - Y$.

a) Find the CDF for Z.

Evidently $F(z) = 0$ for $z \le 0$ and $F(z) = 1$ for $z > 6$. For each $0 < z \le 6$ let $x = \lceil z \rceil \in \{1, 2, 3, 4, 5, 6\}$ be the least integer greater than or equal to z, so $z \in (x - 1, x]$ and

$$\begin{aligned}
\mathsf{P}[Z \le z &= \mathsf{P}[X \le x - 1] + \mathsf{P}[X = x, \ Y > x - z] \\
&= \frac{x-1}{6} + \frac{1}{6}[1 - (x - z)] \\
&= \frac{x - 1 + 1 - x + z}{6} = \frac{z}{6},
\end{aligned}$$

so altogether

$$F(z) = \begin{cases} 0 & -\infty < z \le 0 \\ z/6 & 0 < z \le 6 \\ 1 & 6 < z < \infty \end{cases}$$

and evidently Z has a uniform distribution on $(0, 6)$.

b) Does Z have a density function? If so, find it and identify the distribution. If not, explain why.

Sure. $Z \sim \mathsf{Un}(0, 6)$, so Z has density

$$f(z) = \begin{cases} \frac{1}{6} & 0 < z \le 6 \\ 0 & \text{Otherwise.} \end{cases}$$

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4. Let $X \sim \mathsf{No}(1, 1)$ have a normal distribution with mean $\mu = 1$ and variance $\sigma^2 = 1$. Find:

a) The pdf for X, $f_X(x) =$ _____

$$f_X(x) = \phi(x - 1) = \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}$$

b) The pdf for $Y = |X|$, $f_Y(y) =$ _____

$$\begin{aligned}
f_Y(y) &= f_x(-y) + f_x(y) \\
&= \phi(1 - y) + \phi(1 + y) \\
&= \frac{1}{\sqrt{2\pi}} \left[e^{-(y-1)^2/2} + e^{-(y+1)^2/2} \right]
\end{aligned}$$

for $y > 0$, while $f_Y(y) = 0$ for $y \leq 0$.

c) The pdf for $V = (X - 1)^2$, $f_V(v) =$ _____

$$\begin{aligned}
f_V(v) &= \sum_{\{x: \, g(x)=v\}} \frac{1}{|g'(x)|} f_X(x) \\
&= \frac{1}{|2\sqrt{v}|} f_x(1 - \sqrt{v}) + \frac{1}{|2\sqrt{v}|} f_x(1 + \sqrt{v}) \\
&= \frac{1}{\sqrt{v}} \phi(\sqrt{v}) \\
&= \frac{1}{\sqrt{2\pi v}} e^{-v/2}, \quad v > 0
\end{aligned}$$

for $v > 0$, while $f_V(v) = 0$ for $v \leq 0$. This is the Gamma distribution

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$\mathsf{Ga}(\alpha, \beta)$ with $\alpha = \beta = \frac{1}{2}$. If you prefer the CDF approach,

$$
\begin{aligned}
F_V(v) &= \mathsf{P}[(X - 1)^2 \leq v]; \quad = 0 \text{ for } v < 0; \text{ for } v > 0, \\
&= \mathsf{P}[-\sqrt{v} \leq (X - 1) \leq \sqrt{v}] \\
&= \Phi(v^{1/2}) - \Phi(-v^{1/2}), \text{ since } (X - 1) \sim \mathsf{No}(0, 1), \text{ so} \\
f_V(v) &= \frac{1}{2}v^{-1/2}\phi(\sqrt{v}) + \frac{1}{2}v^{-1/2}\phi(-\sqrt{v}) \\
&= \frac{1}{\sqrt{2\pi v}}e^{-v/2}, \quad v > 0
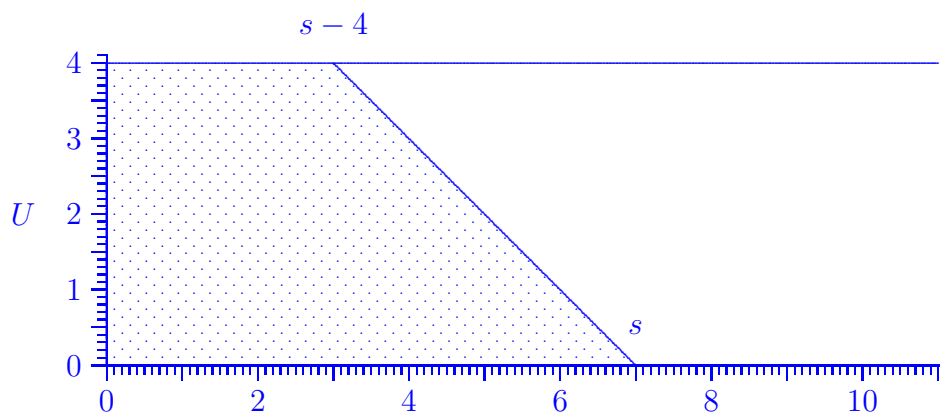\end{aligned}
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5. Chris takes an amount of time (in hours) $U \sim \mathsf{Un}(0, 4)$ to complete probability homework assignments, while Ryan takes an amount of time $X \sim \mathsf{Ex}(0.50)$. Both have mean two hours (*i.e.*, $\mathsf{E}U = 2 = \mathsf{E}X$).

a) If they start at the same time, what is the probability that Chris finishes first?

$$
\begin{aligned}
\mathsf{P}[U < X] &= \int_0^\infty \int_u^\infty f_U(u) f_X(x) dx\, du \\
&= \int_0^4 \int_u^\infty \frac{1}{4}\frac{1}{2}e^{-x/2} dx\, du \\
&= \int_0^4 \frac{1}{4}e^{-u/2} du \\
&= \frac{1}{2}[1 - e^{-2}] \\
&\approx 0.4323324
\end{aligned}
$$

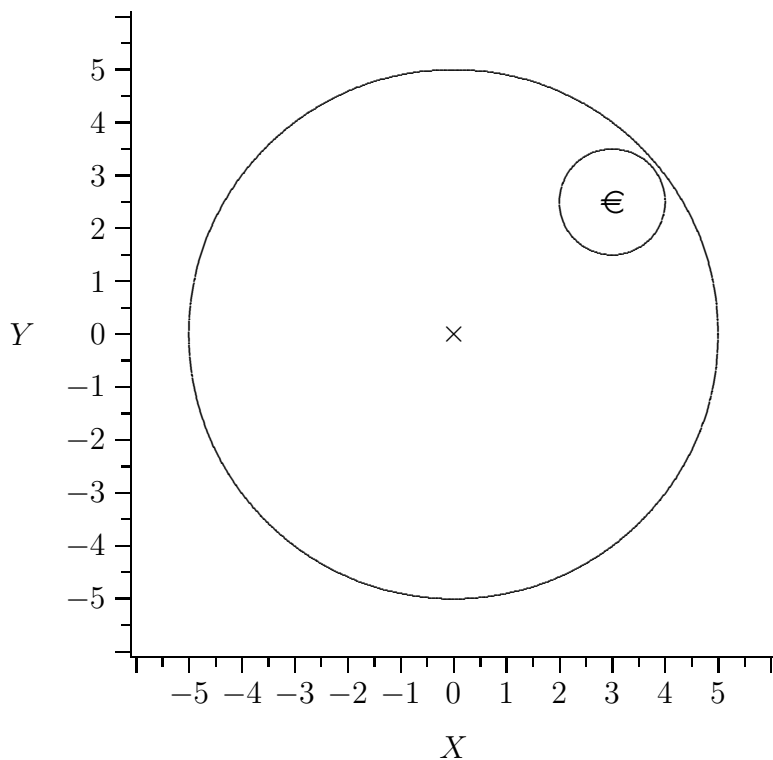b) Find the density function for the total time $S = U + X$ they spend. The limits of integration are easiest to see from a picture:

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The p.d.f. can be calculated directly as:

$$
\begin{aligned}
f_S(s) &= \int_{-\infty}^{\infty} f_U(u)\, f_X(s-u)\, du \\
&= \int_0^s \left(\frac{1}{4}\right)\left(\frac{1}{2}e^{-(s-u)/2}\right) du, \qquad 0 \le s < 4 \\
&= (1 - e^{-s/2})/4 \\
&= \int_0^4 \left(\frac{1}{4}\right)\left(\frac{1}{2}e^{-(s-u)/2}\right) du, \qquad 4 \le s < \infty \\
&= (e^2 - 1)e^{-s/2}/4
\end{aligned}
$$

6.   A Euro coin, with a diameter of 2cm, is dropped into an empty cat food can, whose round bottom has a diameter of 10cm.

a)   What is the probability that the center of the can bottom is covered by the coin?

The coin center must be inside the circle of radius $R \leq 4$, since its *edge* must be no further than 5cm from the can center. The bottom center will be covered by the coin if the coin center lies within $R \leq 1$. Assuming the coin falls uniformly within the allowed circle, the probability of covering the bottom middle will be

$$\mathsf{P}[\text{Center covered by one coin}] = \frac{\pi 1^2}{\pi 4^2} = \frac{1}{16}$$

b)   How many one-Euro coins would you have to drop to have at least a 50% chance of convering the center of the can bottom?

Assuming independence, the probability that the bottom center is *not* covered by n coins would be $\left(\frac{15}{16}\right)^n$; for this to be less than $\frac{1}{2}$ we would need $n \geq \log(1/2)/\log(15/16) \approx 10.74$, so we'd need at least eleven coins. The probability of covering with just ten coins would be $1 - (15/16)^{10} = 0.4755$, the probability of covering with eleven is $1 - (15/16)^{11} = 0.5083$.

If you assume *non-overlapping coins* below, then the probability of covering with n coins would be $n/16$, and eight coins would suffice.

c)   What modeling assumptions did you make to answer parts a) and b) above?

Uniform distribution of the coins and either *independence* (if your answer was 11) or *non-overlapping* (if your answer was 8).