The Compleat Punter

1 Odds at the Race Track

Across the track at Belmont Park stands the flashing odds board. It will have continual updates of the “Morning Line” in the race program you got at the gates.

The “odds” on a horse is the net payout on a $1 ticket for a first-place win by that horse. From the table below, a $1 bet on “Curlin” would get you a $1.20 gain if he wins, and you would pick up the amount $2.20 at the window after the race.

139th Belmont Stakes

<table>
<thead>
<tr>
<th>Post</th>
<th>Horse</th>
<th>Trainer</th>
<th>Jockey</th>
<th>Odds*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Imawildandcrazyguy</td>
<td>Kaplan</td>
<td>Guidry</td>
<td>20-1</td>
</tr>
<tr>
<td>2</td>
<td>Tiago</td>
<td>Shirrexs</td>
<td>Smith</td>
<td>10-1</td>
</tr>
<tr>
<td>3</td>
<td>Curlin</td>
<td>Asmussen</td>
<td>Albarado</td>
<td>6-5</td>
</tr>
<tr>
<td>4</td>
<td>C P West</td>
<td>Zito</td>
<td>Prado</td>
<td>12-1</td>
</tr>
<tr>
<td>5</td>
<td>Slew’s Tizzy</td>
<td>Fox</td>
<td>Bejarano</td>
<td>20-1</td>
</tr>
<tr>
<td>6</td>
<td>Hard Spun</td>
<td>Jones</td>
<td>Gomez</td>
<td>5-2</td>
</tr>
<tr>
<td>7</td>
<td>Rags to Riches</td>
<td>Pletcher</td>
<td>Velazquez</td>
<td>3-1</td>
</tr>
</tbody>
</table>

*Morning-line odds

Belmont Park, New York
1 1/2 mile dirt, left-handed track
3 year olds
purse: U. S. $1 million
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The window payout for a $1 bet on horse $h$ will be the posted odds $r_h$ plus $1$. Also, the window payout on horse $h$ will be the total amount bet on all horses $A$ (the total purse to be distributed) divided by the number of $1$ bets $N_h$ on horse $h$:

$$ r_h + 1 = \frac{A}{N_h}, h = 1, \ldots, 7. $$

So the favorite has the lowest odds, since it has the most backers $N_h$. 
Then summing over the seven runners:

$$\frac{1}{r_1+1} + \ldots + \frac{1}{r_7+1} = \frac{N_1}{A} + \ldots + \frac{N_7}{A} = 1.$$ 

However, this is not quite right, since the track takes some money. The amount distributed is less than the amount bet $A$, hence $r_1 + 1 < \frac{A}{N_1}$ and so in all realistic track betting

$$\sum_{h=1}^{7} \frac{1}{r_h + 1} > 1.$$ 

The sum of reciprocal “odds+1” is $1/(20 + 1) + 1/(10 + 1) + \ldots = 1.25$ at Belmont above.

More generally, when you wager an amount $\$b$ on horse $h$, you pay $\$b$ for the tickets and the net gain when you win is very simply $\$b \times (r_h + 1) - \$b = \$b \times r_h$. When your horse loses, you have lost the $\$b$.

If the sum $\sum \frac{1}{r_h+1}$ were less than 1, you could bet on everything and always come out ahead, a situation called arbitrage. In particular, if you bet amount $\$b$ across all horses with proportion $\frac{1/(r_k+1)}{\sum_k 1/(r_k+1)}$ on horse $h$, the net gain $w$ is the gross payout minus ticket cost $\$b$. If horse 3 wins say (“Curlin”), then you get

$$w = \$b \times \frac{1/(r_3 + 1)}{\sum_k 1/(r_k + 1)} \times (r_3 + 1) - \$b$$

$$= \$b \times \left( \frac{1}{\sum_k 1/(r_k + 1)} - 1 \right)$$

$$> 0.$$ 

Note that the positive gain is the same no matter which horse wins. There is no probability at all in this, it is a money machine. On the other hand, since the opposite inequality holds in reality, the system is a money maker for the track.

2 Odds for Mathematicians

When a probabilist speaks of “odds” then the context is that there is an event that may occur with probability $p$. The event could be “Rags to Riches wins,” and this may happen with probability $p = 1/4$. The odds of success is the ratio $\frac{p}{1-p} = 1/3$.

When we say “odds against” then we mean the reciprocal $\frac{1-p}{p} = 3$.

The odds of Heads in a flip of a fair coin are 1/1, and the odds against a 6 in a roll of a die are 5/1.
3 Payout odds and probabilities

Suppose that nearly the same race for three-year-olds is run repeatedly on tracks of similar lengths and conditions, and Rags to Riches wins a proportion $p_7$ times. Then the law of large numbers implies that repeatedly betting on horse 7 will generate net winnings per race of

$$p_7 \times \$ (r_7 + 1) - \$1.$$  

This is the expected net winnings. That is, on every race you spend $1, and on a fraction $p_7$ you go back to the window and pocket $$($r_7 + 1)$$. If steady betting is a break-even affair, then one would have

$$0 = p_7 \times (r_7 + 1) - 1$$  

giving

$$p_7 = \frac{1}{r_7 + 1}.$$  

Finally, the “odds against horse 7 winning” in this break-even scenario are

$$\frac{1 - p_7}{p_7} = \frac{1 - 1/(r_7 + 1)}{1/(r_7 + 1)} = r_7.$$  

Therefore, we have shown that if the posted odds determined by the betting crowd make for break-even betting in the long run, then the posted payout odds are also the “odds against winning.”

Note however, that with the track’s cut, the posted payout odds will be slightly low. In other words, if horse 7 has payout odds of 3/1 and the posted odds are a reasonable indicator of probabilities, one would expect the true odds against to be higher than the 3/1 payout odds, so typically $p_7 < 1/(3 + 1)$.

If you do not accept the wisdom of the betting crowd and you are sure that horse 7 has a $p_7 = 1/2$ chance of winning, then your expected winnings will be

$$p_7 \times \$(3/1 + 1) - \$1 = \$2 - \$1 = \$1.00.$$  

4 Miscellaneous Wagers

You may hear the following proposition after watching three 6’s come up on a die: “I’ll give you 10-to-1 odds against a 6 next roll.”
This means that the plunger across the table will make you a deal where you will pay him $1 if the next roll is not a 6, and he will pay you $10 if it is a 6. Is this a good deal?

The expected net gain if you accept the deal is

$$10 \times p - 1 \times (1 - p)$$

where $p$ is the probability that a 6 shows on the next roll. This will be at least “break even” (nonnegative) if the probability “odds against” are no greater than the payoff odds:

$$\frac{1 - p}{p} \leq \frac{10}{1}.$$ 

Since consecutive rolls of the die are independent, $p = 1/6$ and the expected net gain is positive:

$$10 \times 1/6 - 1 \times 5/6 = 0.83.$$ 

Payoff odds over 5/1 are going to be a good deal in the long run, and the 10/1 odds are worth taking.