Final Examination

Mth 135 = Sta 104

Wednesday, 2009 December 9, 2:00 – 5:00pm

- This is a **closed book** exam— please put your books on the floor.
- You may use a calculator and **two pages** of your own notes. Do not share calculators or notes.
- Please **ask me** questions if a problem needs clarification.
- **Show your work.** Boxing answers helps me find them.
- Numerical answers: **four significant digits** or fractions **in lowest terms.** Simplify **all** expressions.
- Normal distribution and pdf/pmf tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

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**Signature:**

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**Print Name:**

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<tr>
<td>1.</td>
<td>/20</td>
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<td>4.</td>
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**Total:** /200
Problem 1: Answer the following questions about the events $A$, $B$, $C$:

a) (4) If $A$ and $B$ are independent and each has probability $\frac{2}{3}$, find:

$$P[A \cup B] = \phantom{\text{number}}$$

b) (4) If $P[A \mid B] = \frac{1}{2}$ and $P[B \mid A] = \frac{1}{2}$, what is the largest possible value for $P[A \cap B]$? Give the least upper bound:

$$P[A \cap B] \leq \phantom{\text{number}}$$

c) (4) If $P[A \cup B] = \frac{2}{3}$ and $P[A] = P[B] = \frac{1}{2}$, find:

$$P[A \mid B] = \phantom{\text{number}}$$

d) (4) If $A, B, C$ are independent and each has probability $\frac{1}{3}$, find:

$$P[A \cup B \cup C] = \phantom{\text{number}}$$

e) (4) The events $A$ and $B$ are disjoint. The probability that $B$ will not happen is 0.70, and $P[A \cup B] = 0.5$. Find:

$$P[A] = \phantom{\text{number}}$$
Problem 2: For 4pt each, write your answers in the boxes. Each \( c \) is some unspecified constant. No problem requires (much) integration. Full solutions aren’t required, but show a sketch of the idea. When specifying a “dist’n” you must include any parameter value(s). The second parameter for Normal distribution is variance: No(\( \mu, \sigma^2 \)).

a) If \( X \) has pdf \( f(x) = cx^4(1-x)^6, 0 < x < 1 \), what is \( EX \)?

b) If \( X \sim \text{Un}(-2,10) \), what is the dist’n of \( Y \equiv -X \)?

c) If \( X \sim \text{No}(2,4) \), what is the dist’n of \( Y \equiv -X \)?

d) If \( X \sim \text{Ex}(4) \), what is the dist’n of \( Y \equiv e^{-4X} \)?

e) If \( X \sim \text{Ex}(4) \) and \( Y \sim \text{Ex}(2) \), what is \( E[X + Y] \)?

f) If \( X,Y \) have joint density \( f(x)f(y) \), what is \( P[X > Y] \)?

g) If \( X \sim \text{No}(\mu = 2, \sigma^2 = 4) \), find \( P[X^2 > 4] \).

h) If \( X \sim \text{Po}(4) \), find \( P[X = 0 \mid X \leq 1] \).

i) If \( X,Y \sim \text{No}(1,4) \) are indep, what is \( \text{Cov}[(X + Y), (X - Y)] \)?

j) \( \text{Cov}[X,Y] = 3 \) and \( E[X] = 1 \). What is \( \text{Cov}[X,2Y] \)?
Problem 3: A certain test is going to be repeated until done satisfactorily. Assume that repetitions of the test are independent and that each has probability \( p = 0.30 \) of being satisfactory. Every test performed must be paid for, including the last (satisfactory) one.

a) (5) The Acme Laboratory will perform the tests for $50 each, regardless of the outcome. Find the expected cost \( C_A \) of having Acme run the tests until a satisfactory result is obtained:

\[
E[C_A] = \]

b) (5) Bravo, a competing testing laboratory, also offers the test but with a different pricing schedule: satisfactory tests cost $100, but unsatisfactory tests cost only $25. Find the expected cost \( C_B \) of having Bravo run the tests until a satisfactory result is obtained:

\[
E[C_B] = \]

c) (5) What is the probability that Acme would charge less than Bravo? Show your work.

\[
P[C_A < C_B] = \]

d) (5) All other things being equal, which laboratory would you recommend? Why?

Circle one: A B Reasoning:
Problem 4: The parallel system shown has two components, \(a\) and \(b\); it works as long as either \(a\) or \(b\) continues to function:

![Diagram of parallel system with components \(a\) and \(b\)]

The survival times \(X\) and \(Y\) (in hours) for the two components are independent exponentially-distributed random variables with rates \(2/3\) and \(1/2\) for \(a\) and \(b\), respectively (so \(X\) has density 
\[
f_X(x) = \frac{2}{3} e^{-2x/3} 1_{\{x>0\}}.
\]

a) (10) Find the probability density function for the survival time \(Z\) for the system (first decide how \(Z\) depends on \(X\) and \(Y\)):

\[
f(z) =
\]

b) (10) Find the probability that \(a\) fails before \(b\) does:

\[
P[X < Y] =
\]
Problem 4 (cont):

Recall that $X \sim \text{Ex}(2/3)$ and $Y \sim \text{Ex}(1/2)$ are independent.

c) (10) Find the density function $f(s)$ for the sum $S = X + Y$, correct for all $-\infty < s < \infty$. Simplify! ($S$ does not have a pdf listed on the attached sheet).

$$f(s) =$$

d) (5) What is the probability that both components are still working after two hours? (Hint: this is very easy)

$$P[X > 2 \cap Y > 2] =$$

e) (5) What are the distributions for the survival times $X^*$ and $Y^*$, measured in days? Give either dist’n name with parameter(s), or pdf’s. (Hint: this is also very easy)

$$X^* \sim \quad \quad \quad \quad Y^* \sim \quad \quad \quad \quad$$
Problem 5: A box in the supply room contains a dozen lightbulbs of different ratings: six 40-W, four 60-W, and two 75-W bulbs. All selections and draws are made without replacement.

a) (5) If three are randomly selected, what is the chance they all have the same rating?

b) (5) If three are randomly selected, what is the chance they all have different ratings?

c) (5) If bulbs are drawn successively until a 75-W bulb is found, what is the probability that exactly five draws (including the one for the 75-W bulb) will be required?

d) (5) Find the expected number of 40-W bulbs drawn before any 75-W bulb is drawn:
Problem 6: Let $X$ and $Y$ be normally-distribute random variables

$$ X \sim \text{No}(2, 4) \quad Y \sim \text{No}(1, 25) $$

with variances 4 and 25 and with covariance

$$ \text{Cov}(X, Y) = 6 $$

a) (5) Find the mean and variance of $X - Y$:

$$ E[X - Y] = \quad \text{Var}[X - Y] = \quad $$

b) (5) Find numbers $a, b, c, d, e$ so that we may write

$$ X = a + b Z_1 $$
$$ Y = c + d Z_1 + e Z_2 $$

for independent random variables $Z_1, Z_2 \sim \text{No}(0, 1)$ (Hint: In terms of $a, b, c, d, e$, what are the means, variances, and covariance of $X, Y$?)

c) (5) Find a constant $\phi$ so that $X$ and $Z = (Y - \phi X)$ are independent. Find the mean and variance of $Z$, too.

d) (5) $P[Y \leq 7.5 \mid X = 1] = \quad$

(Hint: Re-write $Y$ in terms of $X$ and either $Z$ from c) or $Z_2$ from b))
Problem 7: Choose the best probability distribution for each random variable below from among the choices Beta, Binomial, Exponential, Gamma, Geometric, Hypergeometric, Negative Binomial, Normal, Poisson, or Uniform (2 pts each) and, whatever the dist’n, give its mean $\mu$ (also 2 pts):

a) The number of test subjects who improve in a clinical trial with 20 participants drawn at random from a population if the treatment is effective for about 20% of subjects:
   ○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
   $\mu =$

b) The number of typographical errors (i.e. mistaken characters) in a 1000 page book, if an average page has about 4000 characters and about 90% of the pages have no errors at all? Assume independence. It is possible for one page to have multiple typographical errors.
   ○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
   $\mu =$

c) The number of jacks in a 13-card hand dealt from a well-shuffled 52-card deck (the entire deck has 4 jacks):
   ○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
   $\mu =$

d) The length of time until 3 East-West campus buses arrive, if about 10 buses arrive per hour and arrivals in disjoint time intervals are independent? Give $\mu$ in minutes.
   ○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
   $\mu =$

e) The number of failures before four successes in an environmental engineering project, with independent trials and 10% success rate?
   ○ Be ○ Bi ○ Ex ○ Ga ○ Ge ○ HG ○ NB ○ No ○ Po ○ Un
   $\mu =$
Problem 8: Let $X \sim \text{Be}(\theta, 1)$ have a Beta $\text{Be}(\alpha, \beta)$ dist’n with parameters $\alpha = \theta$ and $\beta = 1$ for some real number $\theta > 0$. Give requested pdf’s correctly at all real numbers $x$, $y$, etc. Simplify—no answer needs to have a $\Gamma(\cdot)$. Each answer will depend on the value of $\theta > 0$.

a) (2) $f_X(x) = \phantom{\text{E}[X]}$ \hspace{1cm} $E[X] = \phantom{\text{E}[X]}$

Give the probability density function (pdf) and mean for $X$.

b) (6) $f_Y(y) = \phantom{\text{E}[Y]}$ \hspace{1cm} $E[Y] = \phantom{\text{E}[Y]}$

Give the pdf and mean for $Y = X^2$.

c) (6) $f_Z(z) = \phantom{\text{E}[Z]}$ \hspace{1cm} $E[Z] = \phantom{\text{E}[Z]}$

Give the pdf and mean for $Z = -\log X$ (the natural logarithm).

d) (6) $f_R(r) = \phantom{\text{E}[R]}$ \hspace{1cm} $E[R] = \phantom{\text{E}[R]}$

Give the pdf and mean for $R = 1/X$. 

Fall 2009

December 9, 2009
Extra worksheet, if needed:
Table 5.1: Area $\Phi(x)$ under the Standard Normal Curve to the left of $x$. 

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.00</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5000</td>
<td>0.5040</td>
<td>0.5080</td>
<td>0.5120</td>
<td>0.5160</td>
<td>0.5199</td>
<td>0.5239</td>
<td>0.5279</td>
<td>0.5319</td>
<td>0.5359</td>
</tr>
<tr>
<td>1</td>
<td>0.6950</td>
<td>0.6954</td>
<td>0.6958</td>
<td>0.6962</td>
<td>0.6966</td>
<td>0.6969</td>
<td>0.6973</td>
<td>0.6976</td>
<td>0.6979</td>
<td>0.6982</td>
</tr>
<tr>
<td>2</td>
<td>0.8413</td>
<td>0.8446</td>
<td>0.8478</td>
<td>0.8509</td>
<td>0.8531</td>
<td>0.8554</td>
<td>0.8577</td>
<td>0.8599</td>
<td>0.8621</td>
<td>0.8643</td>
</tr>
<tr>
<td>3</td>
<td>0.9973</td>
<td>0.9978</td>
<td>0.9983</td>
<td>0.9987</td>
<td>0.9991</td>
<td>0.9994</td>
<td>0.9997</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

$\Phi(0.6745) = 0.75$  $\Phi(1.4449) = 0.95$  $\Phi(2.3263) = 0.99$  $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$  $\Phi(1.9600) = 0.975$  $\Phi(2.5758) = 0.995$  $\Phi(3.2905) = 0.9995$
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$(\alpha+\beta)^{-1} (\alpha+\beta+1)$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\text{Bi}(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$np$</td>
<td>$(q = 1-p)$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = p q^{x-1}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$(q = 1-p)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f(y) = p q^{y-1}$</td>
<td>$y \in {1, \ldots}$</td>
<td>$1/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>$\text{HG}(n, G, B)$</td>
<td>$f(x) = \frac{\binom{G}{n} \binom{B}{n-x}}{\binom{G+B}{n}}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$nP$</td>
<td>$nP(1-P) \frac{N-n}{N-1}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta (1+e^{-(x-\mu)/\beta})^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2/3$</td>
</tr>
<tr>
<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$</td>
</tr>
<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, p)$</td>
<td>$f(x) = \binom{x+\alpha-1}{\alpha} p^\alpha q^x$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f(y) = \binom{y-1}{\alpha-1} p^\alpha q^{y-\alpha}$</td>
<td>$y \in {\alpha, \ldots}$</td>
<td>$\alpha/p$</td>
<td>$\alpha q/p^2$</td>
</tr>
<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-(x-\mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$\text{Pa}(\alpha, \epsilon)$</td>
<td>$f(x) = \alpha \epsilon^\alpha / x^{\alpha+1}$</td>
<td>$x \in (\epsilon, \infty)$</td>
<td>$\frac{\epsilon \alpha}{\alpha-1}$</td>
<td>$\frac{\epsilon^2 \alpha}{\alpha-1} \frac{\alpha}{(\alpha-1)^2(\alpha-2)}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$\text{Po}(\mu)$</td>
<td>$f(x) = \frac{\mu^x e^{-\mu}}{x!}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>Snedecor $F$</td>
<td>$F(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma((\nu_1+\nu_2)/2)(\nu_1/2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \times$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2-2}$</td>
<td>$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$</td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}} [1 + x^2/\nu]^{-(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$</td>
<td>$\nu/\nu$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$\text{Un}(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \beta)$</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\Gamma(1+\alpha-1)}{\beta^{\alpha-1} \Gamma(1+1/\alpha)}$</td>
<td>$\frac{\Gamma(1+2/\alpha) \Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$</td>
</tr>
</tbody>
</table>