Midterm Examination #1

Mth 136 = Sta 114

Thursday, February 26, 2009
2:50 – 4:05 pm
Version *b*

- This is a **closed book** exam— put your books on the floor.
- You may use a calculator and **one page** of your own notes.
- Do not share calculators or notes.
- **Show** your **work**. Neatness counts. Boxing answers helps.
- Numerical answers: **four significant digits** or fractions **in lowest terms**. Simplify *all* answers.
- Extra worksheet and pdf & distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ____________________

Print **Name**: ____________________

1.	/20
2.	/20
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4.	/20
5.	/20
Total:	/100

Problem 1: Let $\mathbf{x} = \{X_i\}_{i \le n}$ be a simple random sample of n independent random variables with the exponential distribution with rate θ (denoted "$\mathsf{Ex}(\theta)$"), or p.d.f.

$$f(x) = \theta e^{-\theta x}\,\mathbf{1}_{\{x>0\}}.$$

Define statistics

$$S(\mathbf{x}) = \sum_{i \le n} X_i \qquad T(\mathbf{x}) = \min_{i \le n} X_i \qquad V(\mathbf{x}) = \#\{i \le n :\ X_i > 1\}$$

a) (5) Find the Maximum Likelihood Estimator for θ, upon observing $\mathbf{x} = (X_1, ..., X_n)$. Express it in terms of n and one or more of the statistics above, if possible. Show your work.

$\hat{\theta}(\mathbf{x}) =$

b) (5) Find the posterior probability that $\theta > 1$ for a standard exponential prior distribution with density function

$$\xi(\theta) = e^{-\theta}\mathbf{1}_{\{\theta>0\}}$$

if we observe $n = 1$ observation, with value $X_1 = 2$, find:
(4 pts for correct expression, 5 for complete solution).

$\xi(\theta > 1 \mid X_1 = 2) =$

Problem 1 (cont):

The random variables $\{X_i\}_{i\le n}$ are still independent, all with the $\mathsf{Ex}(\theta)$ distribution, with three statistics:

$$S(\mathbf{x}) = \sum_{i\le n} X_i \qquad T(\mathbf{x}) = \min_{i\le n} X_i \qquad V(\mathbf{x}) = \#\{i \le n:\ X_i > 1\}$$

c) (6) For fixed $\theta > 0$ and $n \in \mathbb{N}$, give the probability distribution for each of the three statistics listed above— either by giving its name and parameter(s), or by giving its pdf/pf.

$S \sim$

$T \sim$

$V \sim$

d) (4) Three possible estimators of the quantity $\varphi \equiv \mathsf{P}_\theta[X > 1]$ are:

$$\delta_1(\mathbf{x}) = e^{-n/S(\mathbf{x})} \qquad \delta_2(\mathbf{x}) = e^{-1/nT(\mathbf{x})} \qquad \delta_3(\mathbf{x}) = V(\mathbf{x})/n$$

Which one, if any, is the MLE of φ? ____________________

Which one, if any, is sufficient for θ? ____________________

Very briefly, explain.

Problem 2: The $n = 9$ independent random variables $\{X_i\}_{i \le n}$ are a simple random sample from the Normal distribution $X_i \sim \mathsf{No}(\mu, \sigma^2)$, with variance σ^2 and mean μ. The values of a few statistics from this random sample are:

$$S(\mathbf{x}) = \sum_{i \le n} X_i = 180 \qquad T(\mathbf{x}) = \min_{i \le n} X_i = 1$$

$$V(\mathbf{x}) = \sum_{i \le n} (X_i - \bar{X})^2 = 72 \qquad W(\mathbf{x}) = \text{Median}(\{X_i\}) = 24$$

a) (9) With these data, find a central 98% confidence interval $\big(A(\mathbf{x}), B(\mathbf{x})\big)$ for μ by specifying the numerical values $a = A(\mathbf{x})$ and $b = B(\mathbf{x})$, if the variance parameter is known to be $\sigma^2 = 4$:

$a =$ $b =$

b) (9) Same question, if σ^2 is unknown:

$a =$ $b =$

c) (2) What is the probability that $\mu \in (a,\ b)$ in part b) above? Explain.

Problem 3: For some parameter value $0 < \theta < 1$, the three random variables X, Y, Z have the trinomial distribution with p.f.

$$\mathsf{P}[X = x, Y = y, Z = z \mid \theta] = \frac{10!}{x!\ y!\ z!}(\theta)^{2x}(2\theta[1-\theta])^{y}(1-\theta)^{2z}$$

for non-negative integers x, y, and z **with sum** $\mathbf{x + y + z = 10}$, and zero otherwise. For **one** observation (x, y, z) of the three variables, find:

a) (5) The posterior distribution for θ, with uniform prior distribution $\xi(\theta) = \mathbf{1}_{\{0<\theta<1\}}$ (give the name and any parameter(s); part credit for p.d.f. in simplified form):

$\xi(\theta \mid x, y, z) \sim$

b) (5) The posterior expectation of θ, with uniform prior distribution $\xi(\theta) = \mathbf{1}_{\{0<\theta<1\}}$ (simplify!):

$\mathsf{E}[\theta \mid x, y, z] =$

Problem 3 (cont):

c) (5) The maximum likelihood estimator for θ (simplify!):
$\hat{\theta} =$

d) (5) A real-valued sufficient statistic $T(x, y, z)$ (recall $x + y + z = 10$):
$T(x, y, z) =$

e) (+2) What is the probability distribution for T, given θ?

Problem 4: The volumes (in units of $10^4\,\mathrm{m}^3$) of large volcanic eruptions at the Soufrière Hills Volcano the Caribbean island of Montserrat follow approximately the Pareto distribution with p.d.f.

$$f(v \mid \alpha) = \alpha v^{-\alpha-1} \mathbf{1}_{\{v>1\}}$$

for an unknown parameter $0 < \alpha < 1$.

a) (4) Find the likelihood for n independent observations $\mathbf{v} = \{V_1, \dots, V_n\}$:
$f_n(\mathbf{v} \mid \alpha) =$

b) (4) Find the MLE for α:
$\hat{\alpha}_n(\mathbf{v}) =$

c) (4) For $V \sim f(v \mid \alpha)$ find the moment generating function for $X \equiv \log V$:
$M(t) = \mathsf{E}[e^{tX}] =$

Problem 4 (cont):

d) (4) Find the mean and variance of $X \equiv \log V$ (the MGF may help):

$\mathsf{E}[X] =$ $\mathsf{Var}[X] =$

e) (4) Give the Fisher information for n independent observations:

$I_n(\alpha) =$

Problem 5: The $n = 10$ independent random variables $\{X_i\}_{i\le n}$ are independent, all with the Uniform $\mathsf{Un}(0,\theta)$ probability distribution on the interval $(0,\theta)$ for some unknown parameter $\theta > 0$.

a) (4) Show that the statistic $T(\mathbf{x}) \equiv \max_{i\le n}\{X_i\}$ is sufficient for θ.

b) (4) Find the CDF and pdf for T (they will depend on θ), correctly for *all* $t \in \mathbb{R}$:

$F_\theta(t) =$ $\qquad\qquad$ $f_\theta(t) =$

Problem 5 (cont):

c) (4) Find the squared-error risk of T as an estimator of θ:

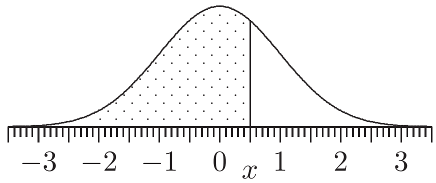
$$R(\theta, T) = \mathsf{E}_\theta |T(\mathbf{x}) - \theta|^2 = \underline{\hspace{4cm}}$$

d) (8) Find an exact 90% symmetric confidence interval $(A(\mathbf{x}), B(\mathbf{x}))$ for θ, by specifying functions $A(\cdot)$ and $B(\cdot)$ with $\mathsf{P}_\theta[A(\mathbf{x}) < \theta < B(\mathbf{x})] = 0.90$ (Hint: A and B will depend only on T):

$A(\mathbf{x}) =$ $\qquad\qquad$ $B(\mathbf{x}) =$

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$



Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

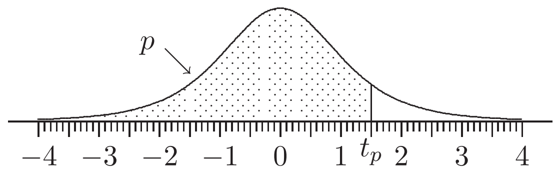
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Critical Values for Student's t

$$p = \int_{-\infty}^{t_p} c \, \frac{dt}{(1+t^2/\nu)^{(\nu+1)/2}}$$



ν	$t_{.60}$	$t_{.70}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$	$t_{.9999}$
1	0.325	0.727	1.376	1.9626	3.078	6.314	12.76	31.82	63.66	318.3	636.6	3183.
2	0.289	0.617	1.061	1.3862	1.886	2.920	4.303	6.965	9.925	22.33	31.60	70.70
3	0.277	0.584	0.978	1.2498	1.638	2.353	3.182	4.541	5.841	10.22	12.92	22.20
4	0.271	0.569	0.941	1.1896	1.533	2.132	2.776	3.747	4.604	7.173	8.610	13.03
5	0.267	0.559	0.920	1.1558	1.476	2.015	2.571	3.365	4.032	5.893	6.869	9.678
6	0.265	0.553	0.906	1.1342	1.440	1.943	2.447	3.143	3.707	5.208	5.959	8.025
7	0.263	0.549	0.896	1.1192	1.415	1.895	2.365	2.998	3.499	4.785	5.408	7.063
8	0.262	0.546	0.889	1.1081	1.397	1.860	2.306	2.896	3.355	4.501	5.041	6.442
9	0.261	0.543	0.883	1.0997	1.383	1.833	2.262	2.821	3.250	4.297	4.781	6.010
10	0.260	0.542	0.879	1.0931	1.372	1.812	2.228	2.764	3.169	4.144	4.587	5.694
11	0.260	0.540	0.876	1.0877	1.363	1.796	2.201	2.718	3.106	4.025	4.437	5.453
12	0.259	0.539	0.873	1.0832	1.356	1.782	2.179	2.681	3.055	3.930	4.318	5.263
13	0.259	0.538	0.870	1.0795	1.350	1.771	2.160	2.650	3.012	3.852	4.221	5.111
14	0.258	0.537	0.868	1.0763	1.345	1.761	2.145	2.624	2.977	3.787	4.140	4.985
15	0.258	0.536	0.866	1.0735	1.341	1.753	2.131	2.602	2.947	3.733	4.073	4.880
16	0.258	0.535	0.865	1.0711	1.337	1.746	2.120	2.583	2.921	3.686	4.015	4.791
17	0.257	0.534	0.863	1.0690	1.333	1.740	2.110	2.567	2.898	3.646	3.965	4.714
18	0.257	0.534	0.862	1.0672	1.330	1.734	2.101	2.552	2.878	3.610	3.922	4.648
19	0.257	0.533	0.861	1.0655	1.328	1.729	2.093	2.539	2.861	3.579	3.883	4.590
20	0.257	0.533	0.860	1.0640	1.325	1.725	2.086	2.528	2.845	3.552	3.85	4.539
21	0.257	0.532	0.859	1.0627	1.323	1.721	2.080	2.518	2.831	3.527	3.819	4.493
22	0.256	0.532	0.858	1.0614	1.321	1.717	2.074	2.508	2.819	3.505	3.792	4.452
23	0.256	0.532	0.858	1.0603	1.319	1.714	2.069	2.500	2.807	3.485	3.768	4.415
24	0.256	0.531	0.857	1.0593	1.318	1.711	2.064	2.492	2.797	3.467	3.745	4.382
25	0.256	0.531	0.856	1.0584	1.316	1.708	2.060	2.485	2.787	3.450	3.725	4.352
26	0.256	0.531	0.856	1.0575	1.315	1.706	2.056	2.479	2.779	3.435	3.707	4.324
27	0.256	0.531	0.855	1.0567	1.314	1.703	2.052	2.473	2.771	3.421	3.690	4.299
28	0.256	0.530	0.855	1.0560	1.313	1.701	2.048	2.467	2.763	3.408	3.674	4.275
29	0.256	0.530	0.854	1.0553	1.311	1.699	2.045	2.462	2.756	3.396	3.659	4.254
30	0.256	0.530	0.854	1.0547	1.310	1.697	2.042	2.457	2.750	3.385	3.646	4.234
40	0.255	0.529	0.851	1.0500	1.303	1.684	2.021	2.423	2.704	3.307	3.551	4.094
60	0.254	0.527	0.848	1.0455	1.296	1.671	2.000	2.390	2.660	3.232	3.460	3.962
120	0.254	0.526	0.845	1.0409	1.289	1.658	1.980	2.358	2.617	3.160	3.373	3.837
∞	0.253	0.524	0.842	1.0364	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.719

Name	**Notation**	**pdf/pmf**	**Range**	**Mean μ**	**Variance σ^2**	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	t_ν	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	

Name: ______________