STA 216, GLM, Lecture 11

October 13, 2009
Reminder: Projects

Hierarchical GLMs

Generalized linear mixed models (GLMMs)
Project

- You should choose a specific project by this Thurs, Oct 15
- Ideally a complex Bayesian analysis involving a regression component, but with a complication relative to standard analyses (missing data, multivariate responses, high-dimensional predictors, informative censoring, nonparametric components, mixture models, etc)
- Do not use something you’ve already done or are well into, but can use a problem or data set related to an RA or collaboration
- Also, appealing to have a project motivated by a scientific question that cannot be addressed by a straightforward analysis.
Hierarchical GLMs

- Extend typical GLMs to incorporate prior distributions for parameters with the GLM
- From frequentist perspective, standard Bayesian GLMs are a type of hierarchical GLM
- Typically, “hierarchical” refer to the incorporation of a shared prior distribution for multiple parameters of a particular type.
- *Hyper-prior* distributions can allow parameters in this prior to be unknown
Mixtures of GLMs

- Poisson log-linear with over-dispersion:

\[(y_i \mid \phi_i, x_i) \sim \text{Poisson}(\phi_i \exp(x_i' \beta))\]
\[\phi_i \sim \text{gamma}(\nu^{-1}, \nu^{-1}),\]

- By adding another level to the Poisson log-linear model, we allow for unexplained heterogeneity among subjects in the Poisson rate parameter.

- The subject-specific parameter, \(\phi_i\), is often referred to as a *frailty* or random effect.

- Subjects with average rate have \(\phi_i = 1\), subjects with above average rate have \(\phi_i > 1\).
Over-Dispersed Poisson Models

- $\phi_i$ can be considered as a *latent variable* summarizing the impact of unmeasured predictors.
- Such unmeasured predictors naturally lead to over-dispersion relative to the typical Poisson log-linear model.
- In particular, marginalizing out $\phi_i$, the mean and variance are:
  
  $$E(y_i \mid x_i) = \exp(x_i' \beta) \quad \text{and} \quad V(y_i \mid x_i) = \exp(x_i' \beta) + \nu \exp(x_i' \beta)^2$$

- Over-dispersion can be introduced in a similar manner in binomial models & more broadly using $\eta_i = a_i + x_i' \beta$, with $a_i \sim N(0, \psi)$. 

Generalized Linear Mixed Model (GLMM)

- GLMM: \( \eta_{ij} = x'_{ij} \beta + z'_{ij} b_i \),
- \( \beta \) are parameters common to all subjects and \( b_i \sim N(0, \Omega) \) are deviations for subject \( i \).
- If we choose \( x_{ij} = z_{ij} \), then all the regression coefficients are assumed to vary for the different study subjects, with the amount of variance dependent on \( \Omega \).
- If we choose \( z_{ij} = 1 \), then only the intercept varies for the different study subjects (random intercept model).
The GLMM is a generalization of the Linear Mixed Effects model of Laird and Ware (1982, *Biometrics*), which is also referred to as a random-effect model.

The Linear Mixed Effects Model has the form,

\[ y_{ij} = x'_{ij} \beta + z'_{ij} b_i + \epsilon_{ij}, \]

where \( \epsilon_{ij} \sim N(0, \sigma^2) \) is an error residual

- \( \beta \) are fixed effects (constant for all subjects)
- \( b_i \sim N(0, \Omega) \) are random effects (varying across subjects)
The Linear Mixed Effects Model can be written as

\[ y_i = X_i \beta + Z_i b_i + \epsilon_i, \]

where \( X_i = (x_{i1}, \ldots, x_{i,n_i})', Z_i = (z_{i1}, \ldots, z_{i,n_i})', \epsilon_i \sim N(0, \Sigma), \) and \( \Sigma = \sigma^2 I_{n_i \times n_i} \)

Hence, \( E(y_i) = X_i \beta + 0 + 0 \) and \( V(y_i) = Z_i \Omega Z_i' + \Sigma \)

So we have

\[ y_i \sim N(X_i \beta, Z_i \Omega Z_i' + \Sigma) \]
Bayesian Analysis for Linear Mixed Model (Gilks et al., 1993)

- **Likelihood Function (Random-intercept case):**

\[
f(y | b_i, \beta, \sigma^2, X) = \prod_{i=1}^{n} \prod_{j=1}^{n_i} (2\pi\sigma^2)^{-1/2} \exp \left\{ -\frac{1}{2\sigma^2} (y_{ij} - x'_{ij}\beta - b_i)^2 \right\},
\]

where \( b_i \sim N(0, \psi^{-1}) \).

- **Conditionally Conjugate Priors:**

\[
\beta \sim N(\beta_0, \Sigma_\beta), \quad \sigma^{-2} \sim G(c_{01}, d_{01}), \quad \text{and} \quad \psi \sim G(c_{02}, d_{02}).
\]
Gibbs Sampler

- Alternately sample from the full conditional posterior distributions of each of the model unknowns

- Full conditional distribution of $\beta$:

$$
\pi(\beta \mid b, \sigma^2, \psi, y, X) \overset{d}{=} N\left(\hat{\Sigma}_\beta \{ \Sigma^{-1}_\beta \beta_0 + \sigma^{-2} \sum_{i=1}^{n} X'_i (y_i - b_i) \}, \hat{\Sigma}_\beta \right),
$$

where $\hat{\Sigma}_\beta = (\Sigma^{-1}_\beta + \sigma^{-2} X'X)^{-1}$.

- Full conditional distribution of $b_i$, for $i = 1, \ldots, n$:

$$
\pi(b_i \mid \beta, \sigma^2, \psi, y, X) \overset{d}{=} N \left( \frac{\sigma^{-2} \sum_{j=1}^{n_i} (y_{ij} - x'_{ij} \beta)}{\psi + \sigma^{-2} n_i}, \frac{1}{\psi + \sigma^{-2} n_i} \right).
$$
Gibbs Sampler

- Full conditional distribution of $\sigma^{-2}$:

$$
\pi(\sigma^{-2} \mid \mathbf{b}, \mathbf{\beta}, \psi, \mathbf{y}, \mathbf{X}) \overset{d}{=} 
\mathcal{G}
\left(c_{01} + \frac{\sum_{i=1}^{n} n_i}{2}, d_{01} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n_i} (y_{ij} - \mathbf{x}_{ij}' \mathbf{\beta} - b_i)^2 \right).
$$

- Full conditional distribution of $\psi$:

$$
\pi(\psi \mid \mathbf{b}, \mathbf{\beta}, \sigma^2, \mathbf{y}, \mathbf{X}) \overset{d}{=} 
\mathcal{G}
\left(c_{02} + \frac{n}{2}, d_{02} + \frac{1}{2} \sum_{i=1}^{n} b_i^2 \right).
$$

- The generalization to multiple random-effects is straightforward, by assuming a conditionally-conjugate Wishart prior for the random-effects precision.
Assignment - Due next Tues, Oct 20th

- To be posted in web notes