Midterm Examination

STA 205: Probability and Measure Theory

Thursday, 2010 Oct 21, 11:40-12:55 pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing please ask me— don’t guess, and don’t discuss questions with others.

Unless a problem states otherwise, you must show your work to get partial credit. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.
Problem 1. Let $(\Omega, \mathcal{F}, P)$ be the unit interval $\Omega = (0, 1]$ with the Borel sets $\mathcal{F} = \mathcal{B}$ and Lebesgue measure $P = \lambda$.

a. (5) Does there exist a random variable $\zeta$ for which $\zeta \in L^p$ for every $0 < p < \infty$, but $\zeta \neq L^\infty$? Find (and verify) one, or prove none exists.
   ○ Yes  ○ No  $\zeta(\omega) =$

b. (15) Let $Z(\omega) = \omega$ and, for $n \in \mathbb{N}$ and $\omega \in \Omega$, set:

   $$X_n = \frac{1}{n} \lfloor nZ \rfloor \quad Y_n = 2^{-n} \lfloor 2^n Z \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to a real number $x$. Prove that:

   i. $Y_n$ increases monotonically to $Z$;
   ii. $X_n \to Z$ but not monotonically.
Problem 2. Let \((\Omega, \mathcal{F}, P)\) be the unit interval \(\Omega = (0, 1]\) with the Borel sets \(\mathcal{F} = \mathcal{B}\) and Lebesgue measure \(P = \lambda\). For each integer \(n \in \mathbb{N}\) set \(X_n(\omega) = n1_{(0,1/n]}(\omega)\).

a. (4) Does \(X_n(\omega)\) converge at each point \(\omega \in \Omega\)? If so, find (and verify) \(X(\omega) \equiv \lim_{n \to \infty} X_n(\omega)\); if not, tell why.

Circle one: Yes No

Reasoning:

b. (4) Does \(\int_{\Omega} |X_n - X| \, dP \to 0\)? Y N
Show why...

c. (4) Does \(\int_{\Omega} |X_n - X|^2 \, dP \to 0\)? Y N
Show why...

d. (4) Is \(\{X_n\}\) uniformly dominated by a positive integrable RV \(Y\)?
If so, find a suitable \(Y\); if not, explain. Y N

e. (4) Is \(\{X_n^2\}\) uniformly dominated by a positive integrable RV \(Y\)?
If so, find a suitable \(Y\); if not, explain. Y N
Problem 3. Let $(\Omega, \mathcal{F}, P)$ be the probability space $\Omega = \{a, b, c, d\}$ with just four points and define a collection of sets by

$$ \mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, \Omega\} $$

a. (5) Is $\mathcal{F}$ a field? If yes, state and illustrate what conditions need to be verified (you don’t have to verify these conditions for every possible combination of sets); if no, give a counter-example.

Circle one:  Yes  No  Reasoning:

b. (5) Define two functions $X, Y : \Omega \to \mathbb{R}$ by

$$ X(a) = 1 \quad X(b) = 2 \quad X(c) = 2 \quad X(d) = 1 \\
Y(a) = 1 \quad Y(b) = 2 \quad Y(c) = 3 \quad Y(d) = 4 $$

Enumerate explicitly the elements of $\sigma(X)$:

$$ \sigma(X) = \left\{ \right\} $$

c. (5) Is $X$ a Random Variable on $(\Omega, \mathcal{F}, P)$?  ○ Yes  ○ No. Why?

d. (5) Is $Y$ a Random Variable on $(\Omega, \mathcal{F}, P)$?  ○ Yes  ○ No. Why?
Problem 4. For \( n \in \mathbb{N} \) let \( U_n \stackrel{iid}{\sim} \text{Un}(0,1) \) be independent uniformly-distributed random variables, and set \( Z_n := -\log(U_n) \).

Give short answers below (in the boxes), with a brief justification:

a. (4) Is \( \{Z_n\} \) uniformly integrable (U.I.)?  
   Why?  
   \( \bigcirc \) Yes  \( \bigcirc \) No

b. (4) \( P[\{Z_n > n\} \text{ for infinitely-many } n] = \)  
   Why?

c. (4) \( P[\{Z_n > \log n\} \text{ for infinitely-many } n] = \)  
   Why?

d. (4) Set \( X_n := \{\min_{1 \leq k \leq n} Z_k\} \); find \( \lim_{n \to \infty} P[n X_n > 1] = \)  
   Why?

e. (4) Set \( Y_n := \{\max_{1 \leq k \leq n} Z_k\} \); find \( \lim_{n \to \infty} P[Y_n \leq 3 + \log n] = \)  
   Why?

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Problem 5. Let \( X_n \) be independent random variables on a probability space \((\Omega, \mathcal{F}, P)\) with common distribution \(\mu(dx)\) on \((\mathbb{R}, \mathcal{B})\) given by
\[
\mu((a, b]) = \frac{1}{a^2} - \frac{1}{b^2}, \quad 1 \leq a < b \leq \infty.
\]
Define
\[
S_n := \sum_{k=1}^{n} X_k \quad \bar{X} := S_n/n.
\]

a. (6) Are the \(\{X_n\}\) Uniformly Integrable? Justify your answer.

b. (6) Are the \(\{\bar{X}_n\}\) Uniformly Integrable? Justify your answer.

c. (8) Does \(\{\bar{X}_n\}\) converge in almost-surely? If so, to what? If not, why?
Problem 6. For 2 pt each, write your answers in the boxes provided, or circle True or False. No explanations are required. In parts a & b, \( g = g(x) \) denotes a completely arbitrary Borel measurable function, not necessarily continuous.

a. If \( X \) is simple\(^1\) and \( Y = g(X) \) then \( Y \) is simple. \( \quad \text{T F} \)

b. If \( X_n \to 0 \) a.s. and \( Y_n = g(X_n) \) then \( Y_n \) converges a.s. \( \quad \text{T F} \)

c. If \( X_n \to 0 \) in \( L_4 \) then \( X_n \to 0 \) in \( L_2 \). \( \quad \text{T F} \)

d. If \( X \) is continuous\(^2\) and \( Y = \cos(X) \) then \( Y \) is continuous. \( \quad \text{T F} \)

e. If \( X \) is abs. cont.\(^3\) and \( Y = 1/X^2 \) then \( Y \) is abs. cont. \( \quad \text{T F} \)

f. If \( E|X_n + 1|^2 \leq 40 \) for each \( n \in \mathbb{N} \) then \( \{X_n\} \) is U.I. \( \quad \text{T F} \)

g. If \( X \in L_2 \) and \( Y \in L_4 \) then for what (if any) \( p \) is \( X \cdot Y \in L_p \)?

h. If \( 0 \leq X \in L_1 \) then \( \sqrt{EX} \geq E[\sqrt{X}] \). \( \quad \text{T F} \)

i. If \( X \in L_2 \) and \( Y_n := n 1_{\{|X|>n\}} \) then \( Y_n \to 0 \) in \( L_2 \) as \( n \to \infty \). \( \quad \text{T F} \)

j. If \( E[X^4] = 96 \), give a non-trivial upper bound for \( P[X > 4] \):

\(^1\)A simple R.V. is one that takes on only finitely-many different values.
\(^2\)This is a sloppy but common way of saying that the random variable has a continuous distribution, i.e., that its CDF \( F(x) \) is a continuous function of \( x \in \mathbb{R} \).
\(^3\)A distribution is absolutely continuous if it is the integral of a density function.
Blank Worksheet
Another Blank Worksheet
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>Be($\alpha, \beta$)</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>Bi($n, p$)</td>
<td>$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$np$</td>
<td>$npq$</td>
</tr>
<tr>
<td>Exponential</td>
<td>Ex($\lambda$)</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>Ga($\alpha, \lambda$)</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>Ge($p$)</td>
<td>$f(x) = p q^{x-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>HG($n, A, B$)</td>
<td>$f(x) = \frac{\binom{A}{x} \binom{B}{n-x}}{\binom{n}{x}}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$n P$</td>
<td>$n P (1-P) \frac{n-n}{n-1}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>Lo($\mu, \beta$)</td>
<td>$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta [1 + e^{-(x-\mu)/\beta}]}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2 / 3$</td>
</tr>
<tr>
<td>Log Normal</td>
<td>LN($\mu, \sigma^2$)</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(x-\mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2} (e\sigma^2 - 1)$</td>
</tr>
<tr>
<td>Neg. Binom.</td>
<td>NB($\alpha, p$)</td>
<td>$f(x) = \binom{\alpha+x-1}{x} p^x q^{\alpha-x}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
</tr>
<tr>
<td>Normal</td>
<td>No($\mu, \sigma^2$)</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(x-\mu)^2/2\sigma^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>Pa($\alpha, \epsilon$)</td>
<td>$f(x) = \frac{1}{\epsilon^\alpha x^{\alpha+1}}$</td>
<td>$x \in (\epsilon, \infty)$</td>
<td>$\frac{\epsilon}{\alpha-1}$</td>
<td>$\frac{\epsilon^2}{(\alpha-1)^2(\alpha-2)}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>Po($\lambda$)</td>
<td>$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Snedecor $F$</td>
<td>$F(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} x^{\frac{\nu_1}{2} - 1} \left[1 + \frac{\nu_2}{\nu_2-x}\right]^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2-2} \left(\frac{\nu_2}{\nu_2-2}\right)^2 2(\nu_1+\nu_2-2) \nu_1(\nu_2-1)$</td>
<td></td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi} \nu^{1/2}} [1 + x^2/\nu]^{-(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$</td>
<td>$\nu/ (\nu - 2)$</td>
</tr>
<tr>
<td>Uniform</td>
<td>Un($a, b$)</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>We($\alpha, \beta$)</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\Gamma(1+\alpha^{-1}) / \beta^{\alpha^{-1}}$</td>
<td>$\Gamma(1+2/\alpha) - \Gamma(1+1/\alpha)$</td>
</tr>
</tbody>
</table>