Sta 205 : Homework 2

1. $\sigma$-Algebras and probability assignments.

(a) Let $\{A, B, C\} \subset \mathcal{F}$ be three events in a probability space $(\Omega, \mathcal{F}, P)$. Recall that a partition is a finite or countable collection of disjoint events $\Lambda_j \in \mathcal{F}$ with $\bigcup \Lambda_j = \Omega$. Enumerate all the elements of the partition $\mathcal{P} = \sigma(P(A, B, C))$ generated by these events (i.e., $\mathcal{P}$ is the smallest partition for which $\{A, B, C\} \subset \sigma(\mathcal{P})$). How many (nonempty) elements does $\mathcal{P}$ have, at most? How many, at minimum?

(b) How many elements does the $\sigma$-algebra $\sigma(\mathcal{P})$ contain? Describe them in words (don’t list them).

(c) Let’s further assume that the above mentioned events $A, B, C$ are disjoint with probabilities $P(A) = 0.6$, $P(B) = 0.3$, $P(C) = 0.1$. Calculate the probability of every event in $\sigma(A, B, C)$.

2. Null sets.

(a) Let $\{A_n, n \in \mathbb{N}\}$ be events such that $P(A_n) = 0$, $\forall n$. Show that $P(\bigcup_{n=1}^\infty A_n) = 0$.

(b) Let $\{B_n, n \in \mathbb{N}\}$ be events such that $P(B_n) = 1$, $\forall n \in \mathbb{N}$. What is $P(\bigcap_{n=1}^\infty B_n)$?

(c) Now consider the set of events, $\{E_\alpha, \alpha \in \mathbb{R}\}$, such that $P(E_\alpha) = 0$, $\forall \alpha \in \mathbb{R}$. Does it necessarily follow that $P(\bigcup_{\alpha \in \mathbb{R}} E_\alpha) = 0$? Give a proof or a counter example.

(d) Finally, let $\{B_k\}$ be a collection of events such that, $\sum_{k=1}^n P(B_k) > n - 1$. Show that $P(\bigcap_{k=1}^n B_k) > 0$.

3. Distribution functions and continuity.

(a) Give an example of a real-valued function on $\mathbb{R}$ which in continuous, but not uniformly continuous.

(b) Let $G$ be a continuous distribution function on $\mathbb{R}$. Show that $G$ is in fact uniformly continuous. Hint: Consider points $\{x_i\}$ for which $G(x_i) = i/n$ for $1 \leq i < n$.

(c) Now let $F$ be any distribution function on $\mathbb{R}$. Show that $F$ can have at most countably many discontinuities. Hint: Consider the open intervals $(F(x-), F(x))$ for discontinuity points $x$. 

1
4. **π & λ - systems.**

(a) Let $\Omega = (0, 1] \times (0, 1]$, and consider the following collections of subsets of $\Omega$:

$$A = \{(0, a] \times (0, b] : 0 < a, b \leq 1\}$$

i. Is $A$ a $\pi$ - system? Why or why not?

ii. Is $A$ a $\lambda$ - system? Why or why not?

(b) Consider the following collecton of subsets of the real line:

$$B = \{(-\infty, b], b \in \mathbb{R}\}$$

i. Show that $B$ is a $\pi$ - system, but not a $\lambda$ system.

ii. What is the $\lambda$ - system generated by $B$? Why?

5. **π - systems and fields.**

(a) Let $\mathcal{C}$ be a non-empty collection of subsets of $\Omega$, and let $\mathcal{F}(\mathcal{C})$ be the minimal field over $\mathcal{C}$. Show that $\mathcal{F}(\mathcal{C})$ consists of sets of the form

$$\bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij},$$

where for each $i, j$ either $A_{ij} \in \mathcal{C}$ or $A_{ij}^c \in \mathcal{C}$, and where the $m$ sets $\bigcap_{j=1}^{n_i} A_{ij}$, $1 \leq i \leq m$, are disjoint. Thus, we can represent the sets in $\mathcal{F}(\mathcal{C})$ explicitly (alas, it turns out that we cannot do the same for the $\sigma$-field $\sigma(\mathcal{C})$).

(b) Now assume further that $\mathcal{C}$ is a $\pi$ system. Show that if $P_1, P_2$ are two probability measures which agree on $\mathcal{C}$, then $P_1, P_2$ must also agree on $\mathcal{F}(\mathcal{C})$. Hint: Use Dynkan’s $\pi$-$\lambda$ theorem, or part(A) and the inclusion-exclusion principle.

(c) Find two probability measures $P_1, P_2$ on some set $\Omega$ that agree on a collection of subsets $\mathcal{C}$, but not on $\mathcal{F}(\mathcal{C})$. Obviously (from (B) above) $\mathcal{C}$ cannot be a $\pi$-system. Hint: It’s enough to have $\mathcal{C} = \{A, B\}$ with just two elements, on an outcome space $\Omega$ with just three points.