1. Expectation.

(a) Let \(X = (X_1, X_2)\) be distributed uniformly over the triangle in \(\mathbb{R}^2\) with vertices \(\{(-1,0), (1,0), (0,1)\}\). Compute \(E(X_1 + X_2)\).

(b) Let \(X \geq 0\) be a random variable on \((\Omega, \mathcal{F}, P)\) and, for \(n \in \mathbb{N}\), set

\[ X_n(\omega) \equiv \min\left(2^n, 2^{-n}[2^n X(\omega)]\right) \]

Prove that \(X_n\) is simple and \(X_n \nearrow X\). Note you must show both monotonicity and convergence. For \(\omega \in \Omega\) and \(\epsilon > 0\), how big must \(n\) be to ensure \(|X - X_n| < \epsilon\)?

(c) Suppose \(X \in L^1(\Omega, \mathcal{F}, P)\), i.e., \(E|X| < \infty\). Show that

\[ \int_{|X| > n} X \, dP \to 0 \quad \text{as} \quad n \to \infty. \]

(d) Let \(\{A_n\}\) denote a sequence of events such that \(P(A_n) \to 0\) as \(n \to \infty\) and let \(X \in L_1\). Show that

\[ \int_{A_n} X \, dP \to 0 \]

(e) Let \(X \in L_1\), and let \(A\) be an event. Show that

\[ \int_A |X| \, dP = 0 \quad \text{iff} \quad P(A \cap [|X| > 0]) = 0 \]

(f) Fix a probability space \((\Omega, \mathcal{F}, P)\) and define a distance measure \(d : \mathcal{F} \times \mathcal{F} \mapsto \mathbb{R}_+\) by \(d(A, B) \equiv P(A \Delta B)\). Show that, if \(\{A_n\} \subset \mathcal{F}\) and \(A \in \mathcal{F}\) satisfy \(d(A_n, A) \to 0\), then

\[ \int_{A_n} X \, dP \to \int_A X \, dP \]

for every \(X \in L^1(\Omega, \mathcal{F}, P)\). Note: Here "\(\Delta\)" denotes the symmetric set difference, \(A \Delta B \equiv (A \setminus B) \cup (B \setminus A)\).
2. Convergence Theorems.

(a) Let $X \geq 0$ be a non-negative random variable. Define sequences of random variables $X_n$ and of extended real numbers $0 \leq S_n \leq \infty$ for positive integers $n \in \mathbb{N}$ by:

$$X_n \equiv \sum_{k=0}^{\infty} \frac{k}{2^n} 1_{\{k < 2^n X \leq k + 1\}}$$

$$S_n \equiv \sum_{k=0}^{\infty} \frac{k}{2^n} P\left[\frac{k}{2^n} < X \leq \frac{k + 1}{2^n}\right]$$

Is $X_n$ “simple”? What is $\lim_{n \to \infty} S_n$? Justify your answer.

(b) Define a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \mathcal{B}((0,1]), \lambda)$ by

$$X_n \equiv \frac{n}{\log n} 1_{(0, \frac{1}{n})} \quad n \in \mathbb{N}.$$ 

Show that $\mathbb{P}[X_n \to 0] = 1$, and that $\mathbb{E}(X_n) \to 0$. Also show that the Dominated Convergence Theorem does not apply to this example. Why?

(c) Suppose $\{Y_n\}$ be a sequence of random variables for $n \in \mathbb{N}$ such that

$$\mathbb{P}(Y_n = \pm n^3) = \frac{1}{2n^2}, \quad \mathbb{P}(Y_n = 0) = 1 - \frac{1}{n^2}$$

Use the Borel-Cantelli lemma to show that $\mathbb{P}[Y_n \to 0] = 1$. Compute $\lim_{n \to \infty} \mathbb{E}(Y_n)$. Is the Dominated Convergence Theorem applicable? Why or why not?

(d) Let $\{X_n\}, X$ be random variables with $0 \leq X_n \to X$. If $\sup_n \mathbb{E}(X_n) \leq K < \infty$, show that $X \in L_1$ and $\mathbb{E}(X) \leq K$. Does $X_n \to X$ in $L_1$?

3. Potpourri.

(a) Let $\{X_n\}$ be a sequence of Bernoulli random variables with

$$\mathbb{P}(X_n = 1) = p_n \quad \mathbb{P}(X_n = 0) = 1 - p_n$$

for some sequence $\{p_n\} \subset [0, 1]$ with $\sum_{n=1}^{\infty} p_n < \infty$. Prove that $\mathbb{P}[X_n \to 0] = 1$. 

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(b) Let \( \{X_n\} \) be a sequence of random variables. Show that

\[
E \left( \sup_{1 \leq n \leq \infty} |X_n| \right) < \infty
\]

if and only if there exists a random variable \( 0 \leq Y \in L_1 \) such that

\[
P(|X_n| \leq Y) = 1, \quad \forall n \in \mathbb{N}.
\]