Sta 205 : Home Work #6

1. Fubini and Tonelli.

(a) Let $X$ be a positive random variable (i.e., $X \geq 0$ a.s). Show that

$$E(X) = \int_0^\infty P(X > t) \, dt$$

(note $X$ need not have an absolutely-continuous distribution). Also verify that for any $\alpha > 0$

$$E(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1} P(X > t) \, dt$$

(b) Define probability spaces $(\Omega_i, \mathcal{F}_i, \mu_i)$, for $i = 1, 2$ as follows. Let each $\Omega_i := (0,1]$, the unit interval, with $\sigma$-algebras

$$\mathcal{F}_1 = \mathcal{B} = \text{Borel sets of } (0,1] \quad \mathcal{F}_2 = 2^{\Omega} = \text{All subsets of } (0,1],$$

and let $\mu_1 = \lambda$ be Lebesgue measure and $\mu_2$ counting measure— so $\mu_1(A)$ is the length of any Borel set $A \in \mathcal{F}_1$ and $\mu_2(B)$ is the cardinality of $B \subset (0,1]$. Define

$$f(x,y) = 1_{x=y}(x,y)$$

Set

$$I_1 = \int_{\Omega_1} \left[ \int_{\Omega_2} f(x,y) \mu_2(dy) \right] \mu_1(dx) \quad I_2 = \int_{\Omega_2} \left[ \int_{\Omega_1} f(x,y) \mu_1(dx) \right] \mu_2(dy)$$

Compute $I_1$ and $I_2$. Is $I_1 = I_2$? Are the measures $\mu_1$ and $\mu_2$ $\sigma$-finite? Why doesn’t Fubini’s theorem hold here?

(c) This problem is a probabilistic version of the familiar integration-by-parts formula from calculus. Suppose $F$ and $G$ are two distribution functions with no common points of discontinuity on an interval $(a,b]$. Show that

$$\int_{(a,b]} G(x)F(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x)G(dx)$$

where “$G(dx)$” denotes the measure on $(\mathbb{R}, \mathcal{B})$ with DF $G(x)$. Show that the formula fails if $F$ and $G$ have common discontinuities.
2. Uniform Integrability (UI).

(a) Let \( \{X_n\} \) be an iid sequence of \( L_1 \) random variables. Set \( S_n \equiv \sum_{i=1}^{n} X_i \). Show that the sequence of random variables \( \{Y_n\} \) defined by \( Y_n \equiv S_n/n \) is UI.

(b) Let \( X_n \sim \text{No}(0, \sigma_n^2) \). Find a simple (easily verifiable) condition on \( \{\sigma_n^2\} \) such that \( \{X_n\} \) is UI.

(c) If \( \{X_n\} \) and \( \{Y_n\} \) are UI, show that so is \( \{X_n + Y_n\} \).

(d) Let \( \{X_n, n \in \mathbb{N}\} \) be an arbitrary sequence of non-negative random variables, and set \( M_n \equiv \vee_{i=1}^{n} X_i \). If \( \{X_n\} \) is UI, show that \( E(M_n)/n \to 0 \).

(e) Let \( \phi(x) \) be a function which grows faster than \( x \) at infinity, i.e., \( \phi(x)/x \to \infty \) as \( x \to \infty \). Let \( C \) be a collection of random variables such that, for some fixed \( B < \infty \) and all \( Z \in C \),

\[
E(\phi(|Z|)) \leq B.
\]

Show that \( C \) is UI. Note: This implies any collection of random variables that is bounded in \( L_p \) for some \( p > 1 \) is UI.

3. Convergence Theorems Revisited.

(a) Let \( X \) be a non-negative real valued random variable. Show that:

i. \( \lim_{n \to \infty} n E(\frac{1}{X}1_{[X>n]}) = 0. \)

ii. \( \lim_{n \to \infty} n^{-1}E(\frac{1}{X}1_{[X>n-1]}) = 0. \)

(b) Let \( \{p_k\} \) be a probability mass function on \( \mathbb{Z}_+ = \{0, 1, \ldots\} \) and define the generating function

\[
P(z) \equiv \sum_{k=0}^{\infty} p_k z^k \quad 0 \leq z \leq 1
\]

Use the Dominated Convergence Theorem to prove that

\[
\frac{d}{dz} P(z) = \sum_{k=1}^{\infty} p_k k z^{k-1} \quad 0 \leq z \leq 1.
\]

Note you may wish to consider the cases \( z < 1 \) and \( z = 1 \) separately. What is \( P'(1) \)? \( P'(0) \)? Can you express the variance of \( X \), if it exists, in terms of \( P(z) \) and its derivatives? Can you find each \( p_k \) explicitly from \( P(z) \)?