1. Convergence In Distribution
   (a) For events \( \{A_n\} \) and \( A \) in some probability space \((\Omega, \mathcal{F}, P)\), define Bernoulli random variables by \( X_n \equiv 1_{A_n} \) and \( X \equiv 1_A \). As \( n \to \infty \),
   i. Under what conditions on \( \{A_n\} \) and \( A \) will \( X_n \Rightarrow X \)?
   ii. Under what conditions on \( \{A_n\} \) and \( A \) will \( X_n \to X \) in \( L_1 \)?
   iii. Under what conditions on \( \{A_n\} \) and \( A \) will \( X_n \to X \) in \( L_\infty \)?
   (b) Let \( \{X_n\} \) be a sequence of R V’s with distributions given by
   \[
   P[X_n = 1 - \frac{1}{n}] = P[X_n = 1 + \frac{1}{n}] = \frac{1}{2}.
   \]
   Show that \( X_n \) converges in distribution, and find the limiting distribution.
   (c) Define probability density functions by
   \[
   f_n(x) = \begin{cases}
   1 - \cos(2n\pi x) & 0 \leq x \leq 1 \\
   0 & \text{otherwise}
   \end{cases}
   \]
   and let \( F_n \) be the corresponding distribution functions. Show that \( F_n \) converges weakly (i.e., in distribution) and find the limit. Also show that the density functions \( f_n \) do not converge pointwise.
   (d) Let \( Y_n \sim \text{No}(\mu_n, \sigma_n^2) \) and \( Y \sim \text{No}(\mu, \sigma^2) \) be normally-distributed random variables. Show that \( Y_n \Rightarrow Y \) if and only if \( \mu_n \to \mu \) and \( \sigma_n^2 \to \sigma^2 \). Find the Kullback-Leibler divergence \( K(\mu \parallel \mu_n) \equiv -\int \log \frac{\mu_n(dx)}{\mu(dx)} \mu(dx) \).

2. Central Limit Theorem (CLT)
   (a) Fix \( a > 1 \) and let \( X_n \) be an iid sequence with density function
   \[
   f(x) = a|x|^{-2a}, \quad |x| \geq 1; \quad f(x) = 0, \quad |x| < 1.
   \]
   Compute \( E[X_1] \) and \( E[X_1^2] \). Set \( S_n \equiv \sum_{i=1}^n X_i \). Find the limiting distributions of \( S_n/n \) and of \( S_n/\sqrt{n} \) as \( n \to \infty \). Extra credit: What happens for \( a < 1 \)? For \( a = 1 \)?
   (b) Delta method. Let \( \{X_j\} \overset{iid}{\sim} \text{Bi}(1, \theta) \) be independent Bernoulli random variables with partial sum \( S_n \equiv \sum_{j=1}^n X_j \sim \text{Bi}(n, \theta) \) and sample mean \( \overline{X}_n \equiv S_n/n \), for some \( \theta \in (0, 1) \), and let \( \phi \in C^\infty(0, 1) \) be an infinitely-differentiable real-valued function on the unit interval. For large \( n \) find the approximate mean and variance of \( \phi(\overline{X}_n) \), correct to order \( 1/n \). Show your work; keep track of the error terms!

\[ ^1 \text{Taylor’s theorem might help} \]