Sta 205 : Homework #11

1. Conditional Expectation

(a) Let \( \{N_t\}_{t \geq 0} \) be a homogeneous Poisson process with rate \( \lambda \), so \( N_0 = 0 \) and for every \( n \in \mathbb{N} \) and \( 0 = t_0 < t_1 < ... < t_n < \infty \) the random variables \( X_i = [N_{t_i} - N_{t_{i-1}}] \) are independent for \( 1 \leq i \leq n \) with \( X_i \sim \text{Po}(\lambda(t_i - t_{i-1})) \) distributions. For \( 0 < s < t < \infty \) find the conditional expectations:

\[
E[N_s \mid N_t] = E[N_t \mid N_s] =
\]

(b) Let \( X_1, X_2 \) be iid unit-rate exponential random variables, \( t > 0 \). Find:

i. \( E[X_1 \mid X_1 + X_2] = \)

ii. \( P[X_1 < 3 \mid X_1 + X_2] = \)

iii. \( E[X_1 \mid X_1 \land t] = \)

iv. \( E[X_1 \mid X_1 \lor t] = \)

(c) Let \( X, Y \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \) and suppose \( E[X \mid Y] = \phi(Y) \) for a monotonically decreasing Borel function \( \phi : \mathbb{R} \to \mathbb{R} \). Prove that \( \text{Cov}(X, Y) \leq 0 \).

(d) If \( X \in L_2(\Omega, \mathcal{F}, \mathbb{P}) \) and \( \mathcal{H} \subset \mathcal{G} \subset \mathcal{F} \), show

\[
E\left[ (X - E[X \mid \mathcal{H}])^2 \right] \geq E\left[ (X - E[X \mid \mathcal{G}])^2 \right]
\]

2. Martingales

(a) Let \( \{(X_n, \mathcal{F}_n), \ n \geq 0\} \) be a martingale on \( (\Omega, \mathcal{F}, \mathbb{P}) \) that is predictable in the sense that \( X_{n+1} \) is \( \mathcal{F}_n \)-measurable for each \( n \). Show that \( X_n = X_0 \) almost-surely.

(b) A sequence \( \{(X_n, \mathcal{F}_n), \ n \geq 0\} \subset L_2(\Omega, \mathcal{F}, \mathbb{P}) \) is a submartingale if \( \mathcal{F}_n \subseteq \mathcal{F}_m \subseteq \mathcal{F} \) and \( X_n \leq E[X_m \mid \mathcal{F}_n] \) for each \( 0 \leq n \leq m < \infty \).

Let \( \{(X_n, \mathcal{F}_n), \ n \geq 0\} \) and \( \{(Y_n, \mathcal{F}_n), \ n \geq 0\} \) be submartingales on \( (\Omega, \mathcal{F}, \mathbb{P}) \). Show that \( X_n \lor Y_n \) and \( X_n + Y_n \) are submartingales too.

(c) For \( 0 < p < 1 \) set \( q = 1 - p \) and let \( \{\xi_j\} \) be iid random variables with \( \mathbb{P}[\xi_j = 1] = p \) and \( \mathbb{P}[\xi_j = -1] = q \). Set

\[
S_n = \sum_{j \leq n} \xi_j,
\]

a biased (unless \( p = 1/2 \)) random walk on the integers.

i. For which \( \alpha \in \mathbb{R} \) is \( S_n - \alpha n \) a martingale?

ii. For which \( \alpha, \beta \in \mathbb{R} \) is \( (S_n)^2 - \alpha S_n - \beta n \) a martingale?

iii. For which \( r > 0 \) is \( r^{S_n} \) a martingale?