Problem 1. Recall that a sequence \( \{x_n\} \) in a metric space \((X, d)\) converges to a limit \(x^* \in X\) if for each \(\epsilon > 0\) there exists a number \(N_\epsilon < \infty\) such that

\[
(\forall n \geq N_\epsilon) \quad d(x_n, x^*) < \epsilon.
\]

(a. Prove\(^1\) that \(x_n = 1/\sqrt{n}\) converges to \(x^* = 0\) in the metric space \(X = \mathbb{R}\) with the usual (Euclidean) distance metric \(d(x, y) \equiv |x - y| = \sqrt{(x - y)^2}\).

(b. Find an explicit sequence \(x_n\) of rational numbers that converges to \(x^* = \pi\) in the metric space \(X = \mathbb{R}\). Prove that it converges, by finding \(N_\epsilon\) (Hint: you might want to start by choosing \(N_\epsilon\)—, say, \([1/\epsilon]\) or \([-\log_2 \epsilon]\) or \([-\log_{10} \epsilon]\)— and then find \(x_n\)).

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\(^1\)Find \(N_\epsilon\) explicitly. You may find the function \(\lfloor x \rfloor \equiv \max\{k \in \mathbb{Z} : k \leq x\}\) (the greatest integer less than or equal to \(x\)) to be useful, or perhaps \(\lceil x \rceil \equiv \min\{k \in \mathbb{Z} : k \geq x\}\).
Problem 2. Recall that a subset $E$ of a metric space $(\mathcal{X}, d)$ is open if for each $x \in E$ there exists some $\epsilon_x > 0$ such that the entire ball

$$B_\epsilon(x) = \{ \xi \in \mathcal{X} : d(x, \xi) < \epsilon_x \} \subset E$$

and that a set $F \subset \mathcal{X}$ is closed if its complement $F^c = \{ x \in \mathcal{X} : x \notin F \}$ is open.

a. Prove that $(0, 1)$ is open in $\mathcal{X} = \mathbb{R}$.

b. Prove that any union $U = \bigcup E_\alpha$ of open sets is also open.

c. Show by example that the union $U = \bigcup F_\alpha$ of closed sets may not be closed.
Problem 3. Recall that a set $K$ in a metric space $(\mathcal{X}, d)$ is \textit{compact} if every open cover $K \subseteq \bigcup_{\alpha} U_\alpha$ admits a finite sub-cover $K \subseteq \bigcup_{i=1}^n U_{\alpha_i}$, and that a function $f(\cdot) : \mathcal{X} \rightarrow \mathcal{Y}$ from one metric space to another is \textit{continuous} if for every open set $U \subseteq \mathcal{Y}$, $f^{-1}(U) = \{x : f(x) \in U\}$ is an open set in $\mathcal{X}$.

a. If $K$ is a \textit{compact} set and $A \subset K$ is a \textit{closed} subset, prove that $A$ is also compact.

b. If $f : \mathcal{X} \rightarrow \mathbb{R}$ is a \textit{continuous} real-valued function and $K \subset \mathcal{X}$ is compact, prove that the supremum

$$M \equiv \sup_{x \in K} f(x)$$

is finite.

c. Show\footnote{Suggestion: take $K = [0,1]$ on $\mathcal{X} = \mathbb{R}$, and define $f(x)$ by cases. What cases?} this can fail if $f$ is not continuous—\textit{i.e.}, give an example of an unbounded (but finite) function $f$ on a compact set $K$. 
Problem 4.

a. Let $K_\alpha$ be compact for each index $\alpha$ and suppose that each finite intersection $\bigcap_{j=1}^n K_{\alpha_j} \neq \emptyset$ is non-empty. Prove that $\cap_\alpha K_\alpha \neq \emptyset$.

b. If $f : \mathcal{X} \to \mathbb{R}$ is real-valued and continuous with supremum $M \equiv \sup_{x \in K} f(x)$ on a compact set $K \subset \mathcal{X}$, prove that there exists some $x^* \in K$ for which $f(x^*) = M$. 
Problem 5.

a. Give an example of a closed set $C \subset \mathbb{R}$ that is not compact.

b. Give an example of a set $A \subset \mathbb{R}$ that is neither closed nor open.

c. Give an example of a set $B \subset \mathbb{R}$ that is both closed and open.
Problem 6. Evaluate the sums and integrals below for every value of $p \in \mathbb{R}$ (some integrals might be infinite or undefined for some values of $p$):

a. $\int_{0}^{1} x^p \, dx =$

b. $\int_{0}^{\infty} e^{-px} \, dx =$

c. $\sum_{n=2}^{9} p^n =$

d. $\sum_{n=1}^{\infty} p^n =$

e. $\int_{0}^{\infty} x \, e^{-px^2} \, dx =$

f. $\int_{0}^{x} \sin(ln \, u) \, du =$

g. $\int_{0}^{\pi} e^{-p \cos(x)} \sin(x) \, dx =$
Problem 7. Which of the following sums and integrals converges? Why?

a. T F \( \int_{2}^{\infty} \frac{\ln(e^x - 2)}{x^3 + 1} \, dx \) converges:

b. T F \( \sum_{n=0}^{\infty} \frac{3^n (n!)^2}{(2n)!} \) converges:

c. T F \( \sum_{n=1}^{\infty} \frac{\ln n + \sin n}{n^{3/2}} \) converges:

d. T F \( \int_{0}^{\infty} \frac{\sin x}{x^{3/2}} \, dx \) converges:

e. T F \( \int_{0}^{\infty} \frac{dx}{\sqrt{x} + x^2} \) converges:

f. T F \( \int_{0}^{1} \frac{\tan x}{x^2} \, dx \) converges: