Final Examination

STA 205: Probability and Measure Theory

Due by Tuesday, 2011 Dec 13, 9:00 am

This is an open-book take-home examination. You may work on it during any consecutive 24-hour period you like; please record your starting and ending times on the lines below.

You must do your own work—no collaboration is permitted. If a question seems ambiguous or confusing please ask me—don’t guess, and don’t discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (684-3275) or, better, by e-mail (wolpert@stat.duke.edu).

You must show your work to get credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible. [Boxing helps.]

This exam is due by 9am Tuesday, 2011 Dec 13. You may slip it under my office door (211c Old Chem) or hand it to me earlier if you wish.

| Print Name: | 1. | /20 |
| Issued: | 8:00 am, Dec 12, 2011 | 2. | /20 |
| Started: | , Dec , 2011 | 3. | /20 |
| Finished: | , Dec , 2011 | 4. | /20 |
| Due by: | 9:00 am, Dec 13, 2011 | 5. | /20 |
| | | 6. | /20 |
| | | | Total: | /120 |
Problem 1: Let $\xi_1, \xi_2, \ldots$ be iid random variables with the Ex(2) distribution (hence mean $E[\xi_j] = 1/2$).

a) Find non-random $a_n \in \mathbb{R}, \ b_n > 0$ such that $S_n \equiv \sum_{1 \leq j \leq n} \xi_j$ satisfies

$$P[(S_n - a_n)/b_n \leq x] \to F(x)$$

for a non-trivial df $F$ (i.e., one for a distribution not concentrated at a single point). Give $a_n, b_n,$ and $F$. Justify your answer.
Problem 1 (cont’d): Still $\{\xi_j\}^{iid} \sim \text{Ex}(2)$.

b) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $X_n \equiv \min_{1 \leq j \leq n} \xi_j$ satisfies:

$$P[(X_n - a_n)/b_n \leq x] \rightarrow G(x)$$

for a non-trivial df $G$. Give $a_n$, $b_n$, and $G$. Justify your answer.

c) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $Y_n \equiv \max_{1 \leq j \leq n} \xi_j$ satisfies

$$P[(Y_n - a_n)/b_n \leq x] \rightarrow H(x)$$

for a non-trivial df $H$. Give $a_n$, $b_n$, and $H$. Justify your answer.
Problem 2: Let \( \{X_n\} \) be a sequence of real-valued random variables on \((\Omega, \mathcal{F}, P)\) such that \( X_n \to Y \) a.s. for some random variable \( Y \).

a) Fix any \( \epsilon > 0 \) and set \( A_n = \{\omega : |X_n(\omega) - Y(\omega)| > \epsilon\} \). Prove that \( P(A_n) \to 0 \) as \( n \to \infty \).

b) Prove that \( \cos(X_n) \to \cos(Y) \) in \( L_1(\Omega, \mathcal{F}, P) \).
Problem 3: Let $A, B, C$ be independent events in $(\Omega, \mathcal{F}, P)$, each with probability strictly above zero and below one.

a) Give explicitly (by enumerating its elements) the smallest $\pi$-system containing $\{A, B, C\}$. How many elements does it have? ______

b) How many elements are in $\sigma \{A, B, C\}$? ______
No need to list them, just tell how many there are.
Problem 3 (cont'd):

c) Construct a random variable $X$ for which $\sigma(X) = \sigma\{A, B, C\}$. Is it possible for $X$ to have a *continuous* distribution?  ○ Yes  ○ No  Why?  
$X =$

d) Now let $\{A_j\} \subset \mathcal{F}$ be an infinite sequence of independent nontrivial events. Construct a random variable $Y$ for which $\sigma(Y) = \sigma\{A_j\}$. Is it possible for $Y$ to have a *continuous* distribution?  ○ Yes  ○ No  Why?  
$Y =$
Problem 4: Let \( \Omega = (0,1)^2 \) be the unit square, with elements denoted \((\omega_1, \omega_2)\), and let \( P \) be Lebesgue measure (area) on the Borel sets \( \mathcal{F} \). For each question below, give the requested random variables \( X, Y \) as explicit functions \( X(\omega) \) and \( Y(\omega) \) for each \( \omega \in \Omega \).

a) Find random variables \( X \) and \( Y \) on \((\Omega, \mathcal{F}, P)\) for such that \( X \) has the \( \text{Ex}(2) \) distribution (exponential, with mean one-half) and \( Y \) has the \( \text{Bi}(1, \frac{1}{2}) \) (Bernoulli, with mean one-half), with \( \sigma(X) \) and \( \sigma(Y) \) independent.

b) Again find random variables \( X \sim \text{Ex}(2) \) and \( Y \sim \text{Bi}(1, \frac{1}{2}) \) on \((\Omega, \mathcal{F}, P), \) but now with \( \sigma(X) \subseteq \sigma(Y) \) if possible (prove it!). If not, explain why.

c) Again find random variables \( X \sim \text{Ex}(2) \) and \( Y \sim \text{Bi}(1, \frac{1}{2}) \) on \((\Omega, \mathcal{F}, P)\), but now with \( \sigma(X) \supset \sigma(Y) \) if possible (prove it!). If not, explain why.
Problem 5: Let \( \{\xi_j\} \) be iid \( \mathbb{E}(1) \) standard exponential random variables. For \( \alpha \in \mathbb{R} \) and \( \beta > 0 \), set \( M_0 = 1 \) and, for \( n \in \mathbb{N} \),

\[
M_n := \beta^{-n} \left\{ \prod_{j=1}^{n} \xi_j \right\}^\alpha
\]

a) For which \( \alpha \in \mathbb{R}, \beta > 0 \) is \( M_n \) a martingale?

b) Find a bound for the probability that \( \left\{ \prod_{j=1}^{n} \xi_j \right\} \) ever exceeds 2.

c) Let \( \alpha = 1 \). For which \( \beta > 0 \) is it certain that \( M_n \geq 2 \) eventually?\(^1\)

Why?

\(^1\) i.e., for which \( \beta \) is it true that \( \mathbb{P}\left[ \sup_{0 \leq n < \infty} M_n \geq 2 \right] = 1? \)
Problem 5 (cont’d): Still \( \{ \xi_j \} \overset{iid}{\sim} \text{Ex}(1) \).

d) Set \( X_j = \xi_j \mathbf{1}_{\{ \xi_j > \log j \}} \). What is the probability that \( \sum X_j \) converges to a finite sum? ________ Why?

e) Set \( Y_j = \xi_j \mathbf{1}_{\{ \xi_j > 2 \log j \}} \). What is the probability that \( \sum Y_j \) converges to a finite sum? ________ Why?
Problem 6: For each part below, select “True” or “False” and give a proof or counter-example to support your answer:

a) T F If \( \{X_j\} \) are \( L_1 \) random variables and \( \sum \|X_j\|_1 < \infty \) then \( S_n = \sum_{1 \leq j \leq n} X_j \) converges in \( L_1 \) to a limit \( S \in \mathcal{L}_1(\Omega, \mathcal{F}, P) \).

b) T F If \( \{X_n\} \), \( Y \) are \( L_2 \) random variables and if \( X_n \to Y \) in \( L_2 \) then \( P[X_n \to Y] = 1 \).

c) T F If \( \{X_n\} \), \( Y \) are \( L_1 \) random variables with \( X_1 \leq 0 \) a.s. and if \( X_n \preceq Y \) decreases to \( Y \) a.s., then \( X_n \to Y \) in \( L_1 \).

d) T F If \( \{X_j\} \) are independent \( L_1 \) random variables with mean \( E[X_j] = 0 \) then \( Y_n = \sum_{1 \leq j \leq n} j^2X_j \) is a martingale.

e) T F If \( X \in L_p \) for every \( 0 < p < \infty \) then also \( X \in L_\infty \).
Problem 6 (cont'd): Provide proofs or counter-examples:

f) T F If \( X, Y \geq 0 \) are independent and neither is in \( L_1 \) then their minimum isn’t in \( L_1 \) either, i.e., \( E[X \wedge Y] = \infty \).

\[ \begin{align*}
g) & \quad T F \quad \text{If } X, Y \geq 0 \text{ and both are in } L_1 \text{ then their maximum is in } L_1 \\
& \quad \text{too, i.e., } E[X \vee Y] < \infty.\end{align*} \]

h) T F If \( X, Y \geq 0 \) and neither is in \( L_1 \) then their product isn’t in \( L_1 \) either, i.e., \( E[XY] = \infty \).

\[ \begin{align*}
i) & \quad T F \quad \text{If } X, Y \geq 0 \text{ are independent and neither is in } L_1 \text{ then their product isn’t in } L_1 \text{ either, i.e., } E[XY] = \infty.\end{align*} \]

\[ \begin{align*}
j) & \quad T F \quad \text{If } X \text{ has a continuous distribution with mean } \mu, \text{ then } Y = 1/X \\
& \quad \text{also has a continuous distribution, with mean } E[Y] = 1/\mu.\end{align*} \]

Phew, you’re done!
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$</td>
<td>$x \in (0,1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\text{Bi}(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$np$</td>
<td>$npq$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}^+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}^+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = p^x (1-p)^{-1}$</td>
<td>$x \in \mathbb{Z}^+$</td>
<td>$q/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f(y) = p q^y (y - 1)$</td>
<td>$y \in {1, \ldots}$</td>
<td>$1/p$</td>
<td>$q/p^2$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>$\text{HG}(n, A, B)$</td>
<td>$f(x) = \frac{\binom{A-1}{x} \binom{n}{A-x}}{\binom{n}{n-A}}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$n P$</td>
<td>$n P (1-P) \frac{n}{x}$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = \frac{\beta^{-1} e^{-\frac{(x-\mu)^2}{\beta^2}}}{\sqrt{2\pi\beta}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2/3$</td>
</tr>
<tr>
<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}^+$</td>
<td>$e^\mu + \sigma^2 / 2$</td>
<td>$e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$</td>
</tr>
<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, p)$</td>
<td>$f(x) = \binom{n-1}{x} p^x (1-p)^{-1}$</td>
<td>$x \in \mathbb{Z}^+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$f(y) = \binom{y-1}{y-\alpha} p^y q^{\alpha-y}$</td>
<td>$y \in {1, \ldots}$</td>
<td>$\alpha/p$</td>
<td>$\alpha q/p^2$</td>
</tr>
<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$\text{Pa}(\alpha, \epsilon)$</td>
<td>$f(x) = \alpha \epsilon^{\alpha} / x^{\alpha+1}$</td>
<td>$x \in (\epsilon, \infty)$</td>
<td>$\epsilon \alpha^{-1}$</td>
<td>$\epsilon^2 \alpha^{-1} (\alpha - 1)$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$\text{Po}(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$x \in \mathbb{Z}^+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Snedecor $F$</td>
<td>$F(v_1, v_2)$</td>
<td>$f(x) = \frac{\Gamma(\frac{v_1}{2}) \Gamma(\frac{v_2}{2})}{\Gamma(\frac{v_1+v_2}{2})} \times \frac{x^{\frac{v_1}{2}}}{1 + \frac{v_1}{v_2} x^{\frac{v_1+2}{2}}}$</td>
<td>$x \in \mathbb{R}^+$</td>
<td>$\frac{v_1}{v_2}$</td>
<td>( \frac{2 [v_2 - 2]}{v_1 [v_2 - 4]} )</td>
</tr>
<tr>
<td>Student $t$</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2}) \sqrt{\pi \nu}} [1 + \frac{x^2}{\nu}]^{-(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$</td>
<td>$\nu / (\nu - 2)$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$\text{Un}(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$a + \frac{\nu}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \beta)$</td>
<td>$f(x) = \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha}$</td>
<td>$x \in \mathbb{R}^+$</td>
<td>$\frac{\Gamma(1+\alpha)}{\beta^{\alpha}}$</td>
<td>$\frac{\Gamma(1+2/\alpha)}{\beta^{\alpha} (1+1/\alpha)}$</td>
</tr>
</tbody>
</table>