Chapter 6: Large Random Samples

Sections

- 6.1: Introduction
- 6.2: The Law of Large Numbers
  - Skip p. 356-358
- 6.3: The Central Limit Theorem
  - Skip p. 366-368
- Skip 6.4: The correction for continuity

Remember: The Midterm is October 25th in class (same room)

- It will cover Chapters 1 - 7
- The exam will be closed book but you may bring a “cheat sheet”
  - One sheet of letter-sized paper. You can write on both sides and no restrictions on what can be on there. I will provide the normal table.
- Homework 7, due Oct. 18 will be from the last part of Chapter 7 and you will get that back on Oct. 23, i.e. before the midterm.
Introduction

- Intuitively we expect the average of many i.i.d. random variables to be close to their mean.

- For example: Let $X_1, X_2, X_3, \ldots$ be a random sample from a $N(\mu, \sigma^2)$ distribution and let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. We can show that for any constant $c$

\[
\lim_{n \to \infty} P(|\bar{X}_n - \mu| \leq c) = 1
\]

- The Law of large numbers gives a mathematical foundation to this for more distributions.

- The Central Limit Theorem gives an approximate probability distribution for how close the sample average is to the mean.
### Inequalities

**Theorem 6.2.1: Markov Inequality**

Let $X$ be a non-negative random variable, i.e. $P(X \geq 0) = 1$. Then for any constant $t > 0$

$$P(X \geq t) \leq \frac{E(X)}{t}$$

- Gives a bound to how much probability can be at large values

**Theorem 6.2.2: Chebychev Inequality**

Let $X$ be a random variable and suppose $\text{Var}(X)$ exists. Then for any constant $t > 0$

$$P(|X - E(X)| \geq t) \leq \frac{\text{Var}(X)}{t^2}$$

- Gives a bound to how far away $X$ is from its mean, and relates it to the variance
Example - Using the Chebychev inequality

- Let $X$ be a continuous random variable with mean $\mu$ and variance $\sigma^2$.
- By using $t = k\sigma$ in the Chebychev inequality we get

$$P(|X - \mu| \geq k\sigma) \leq \frac{\sigma^2}{(k\sigma)^2} = \frac{1}{k^2}$$

which can also be written as

$$P(|X - \mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

In other words: No matter what distribution $X$ has:
- There is at least 75% chance that $X$ is within $2\sigma$ from its mean (set $k = 2$)
- There is at least 88.9% chance that $X$ is within $3\sigma$ from its mean (set $k = 3$)
The sample mean

The *sample mean* is defined as  \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \)

**Theorem 6.2.3: Mean and variance of \( \bar{X} \)**

Let  \( X_1, \ldots, X_n \) be i.i.d. random variables with mean \( \mu \) and variance \( \sigma^2 \). Then

\[
E(\bar{X}_n) = \mu \quad \text{and} \quad \text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}
\]

- That is, the variance of the average is smaller than for a single random variable
- Using Chebychev’s inequality we get (for any distribution)

\[
P(|\bar{X}_n - \mu| \geq t) \leq \frac{\sigma^2}{nt^2}
\]
The weak Law of Large Numbers

**Def: Convergence in probability**

A sequence of random variables, $Z_1, Z_2, Z_3, \ldots$ is said to converge to $b$ in probability if for every number $\epsilon > 0$

$$\lim_{n \to \infty} P(|Z_n - b| < \epsilon) = 1$$

This is often written as

$$Z_n \xrightarrow{P} b$$

**Theorem 6.2.4: (Weak) Law of Large Numbers (LLN)**

Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean $\mu$ and a finite variance. Then

$$\overline{X}_n \xrightarrow{P} \mu$$
Histogram as an approximation to a pdf

**Theorem 6.2.6: Histograms**

Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. random variables. Let $c_1 < c_2$ be two constants. Define $Y_i = 1$ if $c_1 \leq X_i < c_2$ and $Y_i = 0$ otherwise. Then $\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$ is the proportion of $X_i$'s that lie in the interval $[c_1, c_2)$ and

$$\overline{Y}_n \xrightarrow{P} P(c_1 \leq X_1 < c_2)$$

This means that the area of a bar in a histogram converges to the probability of that interval.

I.e. the histogram is an approximation to the pdf.
Example: Random samples from the Beta distribution

Sample size $n = 50$

Sample size $n = 100$

Sample size $n = 1000$

Sample size $n = 5000$
Convergence in distribution

**Def: Convergence in distribution**

Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables. Let $F_n$ be the cdf for $X_n$ for all $n$ and let $F^*$ also be a cdf. We then say that the sequence *converges in distribution to $F^*$* if

$$
\lim_{n \to \infty} F_n(x) = F^*(x)
$$

for all $x$ for which $F^*$ is continuous. $F^*$ is called the *Asymptotic distribution of $X_n$*. 
The Central Limit Theorem

Theorem 6.3.1: Central Limit Theorem (CLT)

Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean $\mu$ and finite variance $\sigma^2$. Then for each fixed number $x$

$$\lim_{n \to \infty} P \left( \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \leq x \right) = \Phi(x)$$

where $\Phi(x)$ is the standard normal cdf.

That is,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}$$

converges in distribution to the standard normal distribution.
Example: Sample mean of Binomials

Say $X_1, X_2, X_3, \ldots$ are i.i.d. Binomial with parameters $k$ and $p$. Then $\mu = E(X_i) = kp$ and $\sigma^2 = \text{Var}(X) = kp(1 - p)$.
For large $n$ the distribution of

$$\frac{\sqrt{n}(\overline{X} - kp)}{\sqrt{kp(1 - p)}}$$

is approximately $N(0, 1)$

In other words, the distribution of the sample mean $\overline{X}_n$ is approximately

$$N\left(kp, \frac{kp(1 - p)}{n}\right)$$

Say $k = 10$, $p = 0.2$ and $n = 25$.
Then the distribution of $\overline{X}_{25}$ is approximately

$$N\left(2, \frac{10 \times 0.2 \times 0.8}{25} \approx 0.064\right)$$
Example: Sample mean of Binomials – continued

The distribution of the sample mean $\bar{X}_n$ is approximately

$$N \left( kp, \frac{kp(1-p)}{n} \right)$$

For $k = 10$, $p = 0.2$ and $n = 25$, the distribution of $\bar{X}_{25}$ is approximately $N(2, 0.064)$

Then we can calculate for example

$$P(\bar{X}_{25} \leq 1.5)$$

We can also calculate the minimum number $n$ so that

$$P(|\bar{X}_n - \mu| < 0.1) \geq 0.9$$
Example: Sampling from a Binomial\((10, 0.2)\) distr.

Histograms of 10,000 sample means and the normal approx. for different \(n\)
Example: Sampling from a Uniform(0, 1) distribution

Histograms of 10,000 sample means and the normal approx. for different $n$
Example: Sampling from an $\text{Expo}(5)$ distribution

Histograms of 10,000 sample means and the normal approx. for different $n$
Example: Empty bottle?

- Suppose that people attending a party pour drinks from a bottle containing 63 ounces of a certain liquid.
- Suppose also that the expected size of each drink is 2 ounces and the standard deviation is 1/2 ounce and that all drinks are poured independently.
- Determine the probability that the bottle will not be empty after 36 drinks have been poured.
Delta method

**Theorem: 6.3.2: Delta Method**

Let $Y_1, Y_2, \ldots$ be a sequence of random variables. Suppose

$$a_n(Y_n - \theta) \text{ converges in distribution to } F^*(x)$$

where $F^*(x)$ is a continuous distribution and $a_1, a_2, \ldots$ is a sequence of numbers such that $\lim_{n \to \infty} a_n = \infty$. Let $g(x)$ be a function with a continuous derivative and $g'(\theta) \neq 0$. Then

$$\frac{a_n(g(Y_n) - g(\theta))}{g'(\theta)} \text{ converges in distribution to } F^*(x)$$
Example: Binomial again

In our Binomial example we found that

$$\frac{\sqrt{n}(\bar{X}_n - 2)}{\sqrt{0.064}}$$

converges in distribution to $N(0, 1)$

Find the asymptotic distribution of $\log(\bar{X}_n)$