Final Examination

STA 205: Probability and Measure Theory

Due by Wednesday, 2010 Dec 15, 9:00 am

This is an open-book take-home examination. You may work on it during any consecutive 24-hour period you like; please record your starting and ending times on the lines below.

You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible.

This exam is due by 9am Wednesday, 2010 Dec 15. You may slip it under my office door (211c Old Chem) or hand it to me earlier if you wish.

Print Name:	____________________
Issued:	7:30 pm, Dec 13 , 2010
Started:	: , Dec , 2010
Finished:	: , Dec , 2010
Due by:	9:00 am, Dec 15 , 2010

1.	/20
2.	/20
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6.	/20
Total:	/120

Problem 1: Let $X_1, X_2, \dots$ be real-valued L_1 random variables on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$[1].

a) If the $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ all have the same probability distribution $\mu = \mathsf{P} \circ {X_n}^{-1}$, show that they are uniformly integrable (UI).

b) If (instead) $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}, \lambda\big)$ and the $\{X_n\}$ satisfy the condition:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall A \in \mathcal{F}) \qquad \mathsf{P}[A] \le \delta \quad \Rightarrow \quad \mathsf{E}\big[\ |X_n| \mathbf{1}_A\ \big] \le \epsilon,$$

then find a uniform bound for $\|X_n\|_1$ (this implies that $\{X_n\}$ are UI).

[1]The conditions on $\{X_n\}$ or $(\Omega, \mathcal{F}, \mathsf{P})$ given in each part of this problem applies *only* to that part.

Problem 1 (cont'd): Still $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$, footnote[1] still applies.

c) If $\mathsf{E}[\ |X_n|^n\ ] \le n$ for $n \in \mathbb{N}$ and $X_n \to 0$ in probability, does it follow that $\mathsf{E}X_n \to 0$? ◯ Yes ◯ No. Give a proof or counterexample.

d) If $X_n = Y_n Z$ with $\|Y_n\|_3 \le B < \infty$ uniformly and $Z \in L_2$ so $A = \|Z\|_2 < \infty$, does it follow that $\{X_n\}$ are UI? ◯ Yes ◯ No
Proof or counter-example:

Problem 2: Let $\Omega = (0, 1]$ with Lebesgue measure $\mathsf{P} = \lambda$ on the Borel sets $\mathcal{F} = \mathcal{B}(\Omega)$, and set $Y(\omega) := 4\omega(1 - \omega)$ for $\omega \in \Omega$.

a) Find the conditional expectation of each arbitrary $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$: $\mathsf{E}[X \mid Y](\omega) =$

b) Find all random variables Z that are independent of $\sigma(Y)$.

Problem 3: Let $\{X_j\}$ be random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ and set $S_n = \sum_{j=1}^n X_j$, their partial sum.

a) (8) If $X_j \to 0$ almost surely, prove that $\frac{1}{n} S_n \to 0$ almost-surely.

b) (6) Give an example where $X_j \to 0$ in probability but $\frac{1}{n} S_n \not\to 0$ in probability.

c) (6) Show that if Ω is **countable** then $X_j \to 0$ almost surely if and only if $X_j \to 0$ in probability (this is a hint for part b) above). Note $\mathcal{F}$ is arbitrary, not necessarily 2^Ω.

Problem 4: Let $\{\xi_n\}$ be iid with any continuous distribution (normal, perhaps) and define the event A_n ("ξ_n is a new record") for $n \in \mathbb{N}$ by[2]

$$A_n = \left\{ \omega : \xi_n > \sup_{1 \le j < n} \xi_j \right\}$$

a) Prove that the event $N = \{\omega : \ \xi_j = \xi_k \text{ for any } j \neq k \ \}$ has probability zero.

b) Find the probability of A_n; show your work.
$\mathsf{P}[A_n] =$

[2]Note $\sup\{\emptyset\} = -\infty$, so $A_1 = \Omega$.

Problem 4 (cont'd): Still $A_n = \{\omega : \xi_n > \sup_{1\le j<n} \xi_j\}$.

c) It's not hard to show by induction (but you don't have to) that the $\{A_n\}$ are independent. Let $X_j = \mathbf{1}_{A_j}$ and $R_n = \sum_{j=1}^n X_j$, the number of new records among the first n observations. Show that $R_n \to \infty$ almost surely, *i.e.*, that no record stands forever.

d) Let $T_k = \inf\{n : \; R_n \ge k\}$ be the time of the k^{th} record. Prove that $\mathsf{E}T_k = \infty$ for all $k \ge 2$, so some records can last a long time (Hint: $T_k \ge T_2$).

Problem 5: Let $\xi_i \overset{\text{iid}}{\sim} \mathsf{No}(\mu, \sigma^2)$ be iid normal random variables with the indicated mean and variance and let $S_n \equiv \sum_{i=1}^{n} \xi_i$ be their partial sum. All martingales below are w.r.t. the filtration $\mathcal{F}_n := \sigma\{\xi_1, \dots, \xi_n\}$.

a) For what numbers $\alpha \in \mathbb{R}$, if any, is

$$M_n^{(1)} := S_n - \alpha\, n$$

a martingale? Show your work.

b) **If** $\mu = 0$, for what $\beta \in \mathbb{R}$, if any, is

$$M_n^{(2)} := (S_n)^2 - \beta\, n$$

a martingale? Show your work.

Problem 5 (cont'd): Still $S_n \equiv \sum_{i=1}^n \xi_i$ with $\xi_i \overset{\text{iid}}{\sim} \mathsf{No}(\mu, \sigma^2)$.

c) For what numbers $(\alpha, \beta) \in \mathbb{R}^2$ do the random variables

$$M_n^{(3)} := \exp\{\alpha S_n - \beta\, n\}$$

form a martingale? Why?

d) For $\mu = 0$ and $\sigma^2 = 1$, and arbitrary $a, b \in \mathbb{R}_+$ use $M_t^{(3)}$ to find the best bound you can that the Gaussian random walk S_t ever[3] crosses the line $a\,t + b$:

$$\mathsf{P}\Big[\sup_{0 \le t \le \infty} (S_t - a\,t - b) \ge 0\Big] \le \underline{\qquad\qquad\qquad\qquad}$$

[3] First bound the probability that it crosses the line before time T; then take the limit as $T \to \infty$.

Problem 6: Let $\Omega = (0, 1]$ be the unit interval.

a) Show that the Borel sigma-algebra $\mathcal{F} = \mathcal{B}(\Omega)$ is *countably generated*, *i.e.*, find a countable collection $\{F_i \in \mathcal{F}\}$ for which $\mathcal{F} = \sigma\{F_i\}$.

b) Let $\mathcal{G} = \sigma\big\{\{\omega\} : \ \omega \in \Omega\big\} \subset \mathcal{F}$ be the σ-algebra generated by the singletons. Show that $\mathcal{G}$ is *not* countably generated (perhaps a surprise, since $\mathcal{G} \subset \mathcal{F}$), *i.e.*, prove that there does not exist a countable collection $\{G_i \in \mathcal{G}\}$ for which $\mathcal{G} = \sigma\{G_i\}$.

c) Let P be an arbitrary (not necessarily absolutely-continuous w.r.t. Lebesgue measure λ) probability measure on $\mathcal{F}$. For each $X \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, describe precisely at least one version of $Y = \mathsf{E}[X \mid \mathcal{G}]$ (extra credit for a full description of *all* versions). Also describe the special case when $\mathsf{P} \ll \lambda$.

Name	**Notation**	**pdf/pmf**	**Range**	**Mean μ**	**Variance σ^2**	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\, e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	