Midterm Examination

STA 205: Probability and Measure Theory

Thursday, 2010 Oct 21, 11:40-12:55 pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
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6.	/20
Total:	/120

Problem 1. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the unit interval $\Omega = (0, 1]$ with the Borel sets $\mathcal{F} = \mathcal{B}$ and Lebesgue measure $\mathsf{P} = \lambda$.

a. (5) Does there exist a random variable ζ for which $\zeta \in L_p$ for every $0 < p < \infty$, but $\zeta \neq L_\infty$? Find (and verify) one, or prove none exists.
◯ Yes ◯ No $\zeta(\omega) =$

b. (15) Let $Z(\omega) = \omega$ and, for $n \in \mathbb{N}$ and $\omega \in \Omega$, set:

$$X_n = \frac{1}{n}\lfloor nZ \rfloor \qquad Y_n = 2^{-n}\lfloor 2^n Z \rfloor.$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to a real number x. Prove that:

i. Y_n increases *monotonically* to Z;

ii. $X_n \to Z$ but *not* monotonically.

Problem 2. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the unit interval $\Omega = (0,1]$ with the Borel sets $\mathcal{F} = \mathcal{B}$ and Lebesgue measure $\mathsf{P} = \lambda$. For each integer $n \in \mathbb{N}$ set $X_n(\omega) = n\,1_{(0,1/n^2]}(\omega)$.

a. (4) Does $X_n(\omega)$ converge at each point $\omega \in \Omega$? If so, find (and verify) $X(\omega) \equiv \lim_{n\to\infty} X_n(\omega)$; if not, tell why.

Circle one: Yes No $X(\omega) =$ ____________________
Reasoning:

b. (4) Does $\int_\Omega |X_n - X|\,dP \to 0$? Y N Show why...

c. (4) Does $\int_\Omega |X_n - X|^2\,dP \to 0$? Y N Show why...

d. (4) Is $\{X_n\}$ uniformly dominated by a positive integrable RV Y? If so, find a suitable Y; if not, explain. Y N

e. (4) Is $\{X_n^2\}$ uniformly dominated by a positive integrable RV Y? If so, find a suitable Y; if not, explain. Y N

Problem 3. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the probability space $\Omega = \{a, b, c, d\}$ with just four points and define a collection of sets by

$$\mathcal{F} = \big\{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, \Omega\big\}$$

a. (5) Is $\mathcal{F}$ a field? If *yes*, state and illustrate what conditions need to be verified (you don't have to verify these conditions for every possible combination of sets); if *no*, give a counter-example.

Circle one: Yes No Reasoning:

b. (5) Define two functions $X, Y : \Omega \to \mathbb{R}$ by

$$\begin{array}{llll} X(a) = 1 & X(b) = 2 & X(c) = 2 & X(d) = 1 \\ Y(a) = 1 & Y(b) = 2 & Y(c) = 3 & Y(d) = 4 \end{array}$$

Enumerate explicitly the elements of $\sigma(X)$:

$$\sigma(X) = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

c. (5) Is X a Random Variable on $(\Omega, \mathcal{F}, \mathsf{P})$? ◯ Yes ◯ No. Why?

d. (5) Is Y a Random Variable on $(\Omega, \mathcal{F}, \mathsf{P})$? ◯ Yes ◯ No. Why?

Problem 4. For $n \in \mathbb{N}$ let $U_n \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ be independent uniformly-distributed random variables, and set $Z_n := -\log(U_n)$.

Give short answers below (in the boxes), **with** a brief **justification**:

a. (4) Is $\{Z_n\}$ uniformly integrable (U.I.)? ◯ Yes ◯ No
Why?

b. (4) $\mathsf{P}\big[\{Z_n > n\} \text{ for infinitely-many } n\ \big] =$ $\boxed{}$
Why?

c. (4) $\mathsf{P}\big[\{Z_n > \log n\} \text{ for infinitely-many } n\ \big] =$ $\boxed{}$
Why?

d. (4) Set $X_n := \big\{\min_{1\le k\le n} Z_k\big\}$; find $\lim_{n\to\infty} \mathsf{P}[n\,X_n > 1] =$ $\boxed{}$
Why?

e. (4) Set $Y_n := \{\max_{1\le k\le n} Z_k\}$; find $\lim_{n\to\infty} \mathsf{P}\big[Y_n \le 3 + \log n\big] =$ $\boxed{}$
Why?

Problem 5. Let X_n be independent random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with common distribution $\mu(dx)$ on $(\mathbb{R}, \mathcal{B})$ given by

$$\mu\big((a,\ b]\big) = 1/a^2 - 1/b^2, \qquad 1 \le a < b \le \infty.$$

Define

$$S_n := \sum_{k=1}^{n} X_k \qquad\qquad \bar{X} := S_n/n.$$

a. (6) Are the $\{X_n\}$ Uniformly Integrable? Justify your answer.

b. (6) Are the $\{\bar{X}_n\}$ Uniformly Integrable? Justify your answer.

c. (8) Does $\{\bar{X}_n\}$ converge in almost-surely? If so, to what? If not, why?

Problem 6. For 2 pt each, write your answers in the boxes provided, or circle True or False. No explanations are required. In parts a & b, $g = g(x)$ denotes a completely arbitrary Borel measureable function, not necessarily continuous.

a. If X is simple[1] and $Y = g(X)$ then Y is simple. T F

b. If $X_n \to 0$ a.s. and $Y_n = g(X_n)$ then Y_n converges a.s. T F

c. If $X_n \to 0$ in L_4 then $X_n \to 0$ in L_2. T F

d. If X is continuous[2] and $Y = \cos(X)$ then Y is continuous. T F

e. If X is abs. cont.[3] and $Y = 1/X^2$ then Y is abs. cont. T F

f. If $\mathsf{E}|X_n + 1|^2 \le 40$ for each $n \in \mathbb{N}$ then $\{X_n\}$ is U.I. T F

g. If $X \in L_2$ and $Y \in L_4$ then for what (if any) p is $X \cdot Y \in L_p$? $\boxed{}$

h. If $0 \le X \in L_1$ then $\sqrt{\mathsf{E}X} \ge \mathsf{E}\big[\sqrt{X}\big]$. T F

i. If $X \in L_2$ and $Y_n := n\,1_{\{|X|>n\}}$ then $Y_n \to 0$ in L_2 as $n \to \infty$. T F

j. If $\mathsf{E}[X^4] = 96$, give a non-trivial upper bound for $\mathsf{P}[X > 4]$: $\boxed{}$

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[1]A *simple* R.V. is one that takes on only finitely-many different values.

[2]This is a sloppy but common way of saying that the random variable has a continuous distribution, *i.e.*, that its CDF $F(x)$ is a continuous function of $x \in \mathbb{R}$.

[3]A distribution is absolutely continuous if it is the integral of a density function.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	