Final Examination

STA 205: Probability and Measure Theory

Due by Tuesday, 2011 Dec 13, 9:00 am

This is an open-book take-home examination. You may work on it during any consecutive 24-hour period you like; please record your starting and ending times on the lines below.

You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (684-3275) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible. Boxing helps.

This exam is due by 9am Tuesday, 2011 Dec 13. You may slip it under my office door (211c Old Chem) or hand it to me earlier if you wish.

Print Name:	____________________
Issued:	8:00 am, Dec 12, 2011
Started:	: , Dec , 2011
Finished:	: , Dec , 2011
Due by:	9:00 am, Dec 13, 2011

1.	/20
2.	/20
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5.	/20
6.	/20
Total:	/120

Problem 1: Let $\xi_1, \xi_2, \dots$ be iid random variables with the $\mathsf{Ex}(2)$ distribution (hence mean $\mathsf{E}[\xi_j] = 1/2$).

a) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $S_n \equiv \sum_{1 \le j \le n} \xi_j$ satisfies

$$\mathsf{P}[(S_n - a_n)/b_n \le x] \to F(x)$$

for a non-trivial df F (*i.e.*, one for a distribution not concentrated at a single point). Give a_n, b_n, and F. Justify your answer.

Problem 1 (cont'd): Still $\{\xi_j\} \stackrel{\text{iid}}{\sim} \mathsf{Ex}(2)$.

b) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $X_n \equiv \min_{1 \le j \le n} \xi_j$ satisfies:

$$\mathsf{P}[(X_n - a_n)/b_n \le x] \to G(x)$$

for a non-trivial df G. Give a_n, b_n, and G. Justify your answer.

c) Find non-random $a_n \in \mathbb{R}$, $b_n > 0$ such that $Y_n \equiv \max_{1 \le j \le n} \xi_j$ satisfies

$$\mathsf{P}[(Y_n - a_n)/b_n \le x] \to H(x)$$

for a non-trivial df H. Give a_n, b_n, and H. Justify your answer.

Problem 2: Let $\{X_n\}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $X_n \to Y$ a.s. for some random variable Y.

a) Fix any $\epsilon > 0$ and set $A_n = \{\omega : \ |X_n(\omega) - Y(\omega)| > \epsilon\}$. Prove that $\mathsf{P}(A_n) \to 0$ as $n \to \infty$.

b) Prove that $\cos(X_n) \to \cos(Y)$ in $L_1(\Omega, \mathcal{F}, \mathsf{P})$.

Problem 3: Let A, B, C be independent events in $(\Omega, \mathcal{F}, \mathsf{P})$, each with probability *strictly* above zero and below one.

a) Give explicitly (by enumerating its elements) the smallest π-system containing $\{A, B, C\}$. How many elements does it have? __________

b) How many elements are in $\sigma\{A, B, C\}$? __________
No need to list them, just tell how many there are.

Problem 3 (cont'd):

c) Construct a random variable X for which $\sigma(X) = \sigma\{A, B, C\}$.
Is it possible for X to have a *continuous* distribution? ◯ Yes ◯ No Why?
$X =$

d) Now let $\{A_j\} \subset \mathcal{F}$ be an infinite sequence of independent nontrivial events. Construct a random variable Y for which $\sigma(Y) = \sigma\{A_j\}$.
Is it possible for Y to have a *continuous* distribution? ◯ Yes ◯ No Why?
$Y =$

Problem 4: Let $\Omega = (0,1]^2$ be the unit square, with elements denoted (ω_1, ω_2), and let P be Lebesgue measure (area) on the Borel sets $\mathcal{F}$. For each question below, give the requested random variables X, Y as explicit functions $X(\omega)$ and $Y(\omega)$ for each $\omega \in \Omega$.

a) Find random variables X and Y on $(\Omega, \mathcal{F}, \mathsf{P})$ for such that X has the $\mathsf{Ex}(2)$ distribution (exponential, with mean one-half) and Y has the $\mathsf{Bi}(1, \frac{1}{2})$ (Bernoulli, with mean one-half), with $\sigma(X)$ and $\sigma(Y)$ independent.

b) Again find random variables $X \sim \mathsf{Ex}(2)$ and $Y \sim \mathsf{Bi}(1, \frac{1}{2})$ on $(\Omega, \mathcal{F}, \mathsf{P})$, but now with $\sigma(X) \subset \sigma(Y)$ if possible (prove it!). If not, explain why.

c) Again find random variables $X \sim \mathsf{Ex}(2)$ and $Y \sim \mathsf{Bi}(1, \frac{1}{2})$ on $(\Omega, \mathcal{F}, \mathsf{P})$, but now with $\sigma(X) \supset \sigma(Y)$ if possible (prove it!). If not, explain why.

Problem 5: Let $\{\xi_j\}$ be iid $\mathsf{Ex}(1)$ standard exponential random variables. For $\alpha \in \mathbb{R}$ and $\beta > 0$, set $M_0 = 1$ and, for $n \in \mathbb{N}$,

$$M_n := \beta^{-n} \left\{ \prod_{j=1}^{n} \xi_j \right\}^{\alpha}$$

a) For which $\alpha \in \mathbb{R}$, $\beta > 0$ is M_n a martingale?

b) Find a bound for the probability that $\left\{ \prod_{j=1}^{n} \xi_j \right\}$ *ever* exceeds 2.

c) Let $\alpha = 1$. For which $\beta > 0$ is it certain that $M_n \geq 2$ eventually?[1] Why?

[1] *i.e.*, for which β is it true that $\mathsf{P}\Big[\sup_{0 \leq n < \infty} M_n \geq 2 \Big] = 1$?

Problem 5 (cont'd): Still $\{\xi_j\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$.

d) Set $X_j = \xi_j \mathbf{1}_{\{\xi_j > \log j\}}$. What is the probability that $\sum X_j$ converges to a finite sum? __________ Why?

e) Set $Y_j = \xi_j \mathbf{1}_{\{\xi_j > 2\log j\}}$. What is the probability that $\sum Y_j$ converges to a finite sum? __________ Why?

Problem 6: For each part below, select "True" or "False" and give a proof or counter-example to support your answer:

a) T F If $\{X_j\}$ are L_1 random variables and $\sum \|X_j\|_1 < \infty$ then $S_n = \sum_{1\le j\le n} X_j$ converges in L_1 to a limit $S \in \mathrm{L}_1(\Omega, \mathcal{F}, \mathsf{P})$.

b) T F If $\{X_n\}$, Y are L_2 random variables and if $X_n \to Y$ in L_2 then $\mathsf{P}[X_n \to Y] = 1$.

c) T F If $\{X_n\}$, Y are L_1 random variables with $X_1 \le 0$ a.s. and if $X_n \searrow Y$ decreases to Y a.s., then $X_n \to Y$ in L_1.

d) T F If $\{X_j\}$ are independent L_1 random variables with mean $\mathsf{E}[X_j] = 0$ then $Y_n = \sum_{1\le j\le n} j^2 X_j$ is a martingale.

e) T F If $X \in L_p$ for every $0 < p < \infty$ then also $X \in L_\infty$.

Problem 6 (cont'd): Provide proofs or counter-examples:

f) T F If $X, Y \geq 0$ are independent and *neither* is in L_1 then their minimum isn't in L_1 either, *i.e.*, $\mathsf{E}[X \wedge Y] = \infty$.

g) T F If $X, Y \geq 0$ and *both* are in L_1 then their maximum is in L_1 too, *i.e.*, $\mathsf{E}[X \vee Y] < \infty$.

h) T F If $X, Y \geq 0$ and *neither* is in L_1 then their product isn't in L_1 either, *i.e.*, $\mathsf{E}[X\,Y] = \infty$.

i) T F If $X, Y \geq 0$ are *independent* and *neither* is in L_1 then their product isn't in L_1 either, *i.e.*, $\mathsf{E}[X\,Y] = \infty$.

j) T F If X has a continuous distribution with mean μ, then $Y = 1/X$ also has a continuous distribution, with mean $\mathsf{E}[Y] = 1/\mu$.

Phew, you're done!

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	