Midterm Examination

STA 205: Probability and Measure Theory

Thursday, 2011 Oct 20, 11:40-12:55 pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.

Print Name: ____________________

1.	/20
2.	/20
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4.	/20
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6.	/20
Total:	/120

Problem 1. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the unit interval $\Omega = (0, 1]$ with the Borel sets $\mathcal{F} = \mathcal{B}$ and Lebesgue measure $\mathsf{P} = \lambda$. For $\omega \in \Omega$, set:

$$X_n = \frac{2^n}{\lceil 2^n \omega \rceil} \qquad Y = \frac{1}{\omega}.$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to a real number x— so, for example, $X_2 = 1, 4/3, 2, 4$ with equal probabilities.

a. (5) Is $X_n \in L_1$ for every $n \in \mathbb{N}$? ◯ Yes ◯ No. Prove it.[1]

b. (5) Is $Y \in L_1$? ◯ Yes ◯ No. Prove it.

[1] Hint: This is *easy* and doesn't require (much) calculation.

Problem 1 (cont).

Still

$$X_n = \frac{2^n}{\lceil 2^n \omega \rceil} \qquad Y = \frac{1}{\omega}.$$

c. (5) Is the distribution of $X_n(\omega)$: ◯ Absolutely Continuous, ◯ Singular Continuous, ◯ Discrete, or ◯ None of these? Why?

d. (5) You don't have to show it but, for each $\omega \in \Omega$, $X_n(\omega)$ increases monotonically to Y as $n \to \infty$. What does the Monotone Convergence Theorem say about the expectations of X_n? (For full credit, be specific— use what you found in parts (a) and (b))

Problem 2. Define a function $F(x)$ on $\mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}_{\{x \ge n\}}$$

a. (4) Show that F is a distribution function (or "df").

b. (4) If X is a random variable with distribution function F, find

$\mathsf{P}[X \text{ is even }] =$ $\qquad\qquad$ $\mathsf{P}[X > \pi] =$

Problem 2 (cont).

Recall that $L_p \equiv \{X : \mathsf{E}[\,|X|^p\,] < \infty\}$ for *all* $p > 0$. Let X be a random variable with df $F(x) \equiv \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}_{\{x \geq n\}}$, $x \in \mathbb{R}$.

c. (6) For which $0 < p < \infty$ is X in L_p? Why?

$X \in L_p$ for ____ $< p <$ ____

d. (6) Set $Y := \exp(X)$. For which $0 < p < \infty$ is Y in L_p? Why?

$Y \in L_p$ for ____ $< p <$ ____

Problem 3. Let $\Omega = \mathbb{Z}$ be the set of all integers $-\infty < n < \infty$. Denote by $\#(A)$ the number of elements of any set $A \subset \Omega$ (so $0 \le \#(A) \le \infty$) and let $\mathcal{A} = \{A \subset \Omega : \ \#(A) < \infty\} \subset 2^\Omega$ be just the **finite** sets of integers.

a. (5) Show that $\mathcal{A}$ is a π-system.

b. (10) Show that the smallest λ-system containing $\mathcal{A}$ is all of 2^Ω— *i.e.*, show that *every* subset $B \subset \Omega$ is in $\lambda(\mathcal{A})$. Be explicit.

c. (5) Let $P = \{2, 3, 5, 7, 11, 13, ...\} \subset \Omega$ be the set of all prime numbers, and let $\mathcal{L} = \lambda(P)$ be the smallest λ-system containing P. Give $\mathcal{L}$ explicitly. How many elements does it have? Is it a field?

Problem 4. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the positive half-line $\Omega = \mathbb{R}_+$ with the Borel sets $\mathcal{F} = \mathcal{B}(\Omega)$ and for $A \in \mathcal{F}$ let $\mathsf{P}(A) = \int_A e^{-\omega}\, d\omega$. For each integer $n \in \mathbb{N}$ set

$$X_n(\omega) = n\, \mathbf{1}_{\{[n,\infty)\}}(\omega) = \begin{cases} 0 & \omega < n \\ n & \omega \geq n \end{cases}.$$

a. (6) Does $X_n(\omega)$ converge at each point $\omega \in \Omega$? Y N
If "Yes", what is the limit $X(\omega) \equiv \lim_{n\to\infty} X_n(\omega)$?

$X(\omega) =$ ____________________ If "No", tell why:

b. (6) Set $Z = \sum_{n\in\mathbb{N}} X_n$. Is Z a.s. Finite? Y N Bounded? Y N

c. (8) Is $\{X_n\}$ uniformly dominated by a positive integrable RV Y?
If so, find a suitable Y; if not, explain. Y N

Problem 5. Let $\{X_i : \ 1 \le i \le n\}$ be random variables on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and set $X := \prod_{i=1}^n X_i$.

a. (5) Give an example of $(\Omega, \mathcal{F}, \mathsf{P})$, n, and $\{X_i : \ 1 \le i \le n\}$ where each $X_i \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ but their product $X \notin L_1(\Omega, \mathcal{F}, \mathsf{P})$:[2]

b. (5) If $\{X_i\}$ are independent, and $t > 0$ is any positive real number, are the random variables $\left\{X_i \mathbf{1}_{\{|X_i| \le t\}}\right\}$ independent too? Why?

[2]Hint: $n = 2$ is enough

Problem 5 (cont).

Still $\{X_i : 1 \le i \le n\}$ are random variables and $X := \prod_{i=1}^n X_i$.

c. (5) What theorem allows us to conclude that[3]

$$\lim_{t\to\infty} \mathsf{E}\left[\prod_{i=1}^{n} \big(|X_i| \wedge t\big)\right] = \mathsf{E}\left[\prod_{i=1}^{n} |X_i|\right]$$

d. (5) If each $X_i \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, and if $\{X_i\}$ are independent, complete the proof that their product $X = \prod_{i=1}^n X_i$ is in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, too.

[3]The notation "$a \wedge b$" denotes the *minimum* of real numbers a, b.

Problem 6. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be $\Omega = (0, 1]$, the unit interval, with Borel sets $\mathcal{F}$ and Lebesgue measure P, and let X, Y, and $\{X_n\}$ all be random variables on $(\Omega, \mathcal{F}, \mathsf{P})$. Suppose $X_n(\omega) \to X(\omega)$ as $n \to \infty$ for each $\omega \in \Omega$, and let $\{A_n\} \subset \mathcal{F}$ be events. For each statement below circle **T** for True or **F** for False and, if true, indicate which applies: Fatou's Lemma (Fat), Lebesgue's dominated (DCT) or monotone (MCT) convergence theorems, the Borel-Cantelli (B/C) Lemma (either half), or the inequalities named after Jensen, Minkowski, Hölder, or Markov. **No explanations** are needed— but if you've forgotten the name of a result or inequality, or find a question ambiguous, you can use the space provided.

a. (4) T F If $\mathsf{E}[X_n^2 - 1] > 0$, then $\{X_n\}$ is UI.
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

b. (4) T F If $|X_n| \le 10$ a.s., then $\lim \mathsf{E}X_n = \mathsf{E}X$.
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

c. (4) T F If $\mathsf{E}|X_n| \le 10$, then $\lim \mathsf{E}X_n = \mathsf{E}X$.
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

d. (4) T F If $\mathsf{E}|X_n|^2 \le 100$, then $(\forall t > 0)\ \mathsf{P}[|X_n| > t] < 10/t$.
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

e. (4) T F If $X \in L_1$, then $\mathsf{E}X \le \log \mathsf{E}e^X$
◯ Fat ◯ DCT ◯ MCT ◯ B/C ◯ Jen ◯ Min ◯ Höl ◯ Mar

Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	