STA 711: Probability and Measure Theory
Analysis & Calculus Quiz

Students in STA 711: Probability & Measure Theory are expected to be familiar with real analysis at an advanced undergraduate level—the level of W. Rudin’s Principles of Mathematical Analysis or M. Reed’s Fundamental Ideas of Analysis. They should be able to answer the questions in this quiz without consulting reference materials.

Problem 1. Recall that a sequence \( \{x_n\} \) in a metric space \((\mathcal{X}, d)\) converges to a limit \(x^* \in \mathcal{X}\) if for each \(\epsilon > 0\) there exists a number \(N_\epsilon < \infty\) such that

\[
(\forall n \geq N_\epsilon) \quad d(x_n, x^*) < \epsilon.
\]

a. Prove\(^1\) that \(x_n \equiv 1/\sqrt{n}\) converges to \(x^* = 0\) in the metric space \(\mathcal{X} = \mathbb{R}\) with the usual (Euclidean) distance metric \(d(x, y) \equiv |x - y| = \sqrt{(x - y)^2}\).

b. Find an explicit sequence \(x_n\) of rational numbers that converges to \(x^* = \pi\) in the metric space \(\mathcal{X} = \mathbb{R}\). Prove that it converges, by finding \(N_\epsilon\) (Hint: you might want to start by choosing \(N_\epsilon\)—say, \([1/\epsilon]\) or \([[-\log_2 \epsilon]\) or \([[-\log_{10} \epsilon]\) —and then find \(x_n\)).

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\(^1\)Find \(N_\epsilon\) explicitly. You may find the function \(\lfloor x \rfloor \equiv \max\{k \in \mathbb{Z} : k \leq x\}\) (the greatest integer less than or equal to \(x\)) to be useful, or perhaps \(\lceil x \rceil \equiv \min\{k \in \mathbb{Z} : k \geq x\}\).
Problem 2. Recall that a subset $E$ of a metric space $(\mathcal{X}, d)$ is open if for each $x \in E$ there exists some $\epsilon_x > 0$ such that the entire ball

$$B_\epsilon(x) = \{ \xi \in \mathcal{X} : d(x, \xi) < \epsilon_x \} \subset E$$

and that a set $F \subset \mathcal{X}$ is closed if its complement $F^c = \{ x \in \mathcal{X} : x \notin F \}$ is open.

a. Prove that $(0,1)$ is open in $\mathcal{X} = \mathbb{R}$.

b. Prove that any union $U = \bigcup E_\alpha$ of open sets is also open.

c. Show by example that the union $U = \bigcup F_\alpha$ of closed sets may not be closed.
Problem 3. Recall that a set $K$ in a metric space $(X, d)$ is compact if every open cover $K \subseteq \bigcup_{\alpha} U_{\alpha}$ admits a finite sub-cover $K \subseteq \bigcup_{i=1}^{n} U_{\alpha}$, and that a function $f(\cdot) : X \to Y$ from one metric space to another is continuous if for every open set $U \subseteq Y$, $f^{-1}(U) = \{x : f(x) \in U\}$ is an open set in $X$.

a. If $K$ is a compact set and $A \subseteq K$ is a closed subset, prove that $A$ is also compact.

b. If $f : X \to \mathbb{R}$ is a continuous real-valued function and $K \subseteq X$ is compact, prove that the supremum

$$M \equiv \sup_{x \in K} f(x)$$

is finite.

c. Show\(^2\) this can fail if $f$ is not continuous— i.e., give an example of an unbounded (but finite) function $f$ on a compact set $K$.

\(^2\)Suggestion: take $K = [0, 1]$ on $X = \mathbb{R}$, and define $f(x)$ by cases. What cases?
Problem 4.

a. Let $K_\alpha$ be compact for each index $\alpha$ and suppose that each finite intersection $\bigcap_{j=1}^n K_{\alpha_j} \neq \emptyset$ is non-empty. Prove that $\cap_{\alpha} K_{\alpha} \neq \emptyset$.

b. If $f : \mathcal{X} \to \mathbb{R}$ is real-valued and continuous with supremum $M \equiv \sup_{x \in K} f(x)$ on a compact set $K \subset \mathcal{X}$, prove that there exists some $x^* \in K$ for which $f(x^*) = M$. 
Problem 5.

a. Give an example of a closed set $C \subset \mathbb{R}$ that is not compact.

b. Give an example of a set $A \subset \mathbb{R}$ that is neither closed nor open.

c. Give an example of a set $B \subset \mathbb{R}$ that is both closed and open.
Problem 6. Evaluate the sums and integrals below for every value of $p \in \mathbb{R}$ (some expressions might be infinite or undefined for some values of $p$):

a. $\int_0^1 x^p \, dx =$

b. $\int_0^\infty e^{-px} \, dx =$

c. $\sum_{n=2}^9 p^n =$

d. $\sum_{n=1}^\infty p^n =$

e. $\sum_{n=7}^\infty n p^n =$

f. $\int_0^\infty x e^{-px^2} \, dx =$

g. $\int_0^x \sin(\ln u) \, du =$

h. $\int_0^\infty e^{-p\cos(x)} \sin(x) \, dx =$
Problem 7. Which of the following sums and integrals converges? Why?

a. T F \( \int_{2}^{\infty} \frac{\ln(e^x - 2)}{x^3 + 1} \, dx \) converges:

b. T F \( \sum_{n=0}^{\infty} \frac{3^n (n!)^2}{(2n)!} \) converges:

c. T F \( \sum_{n=1}^{\infty} \frac{\ln n + \sin n}{n^{3/2}} \) converges:

d. T F \( \int_{0}^{\infty} \frac{\sin x}{x^{3/2}} \, dx \) converges:

e. T F \( \int_{0}^{\infty} \frac{dx}{\sqrt{x} + x^2} \) converges:

f. T F \( \int_{0}^{1} \frac{\tan x}{x^2} \, dx \) converges: