Final Examination

STA 711: Probability & Measure Theory

Wednesday, 2012 Dec 12, 9:00 am – 12:00 n

This is a closed-book examination. You may use a sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing, please ask me to clarify it.

Unless a problem states otherwise, you must show your work. There are blank worksheets and a pdf/pmf sheet at the end of the test. It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find. Good luck.

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Print Name: __________________________
**Problem 1.** Let $\mathcal{A}$ be a collection of subsets of a nonempty set $\Omega$ such that

a. $\Omega \in \mathcal{A}$

b. $A, B \in \mathcal{A} \Rightarrow A \setminus B = A \cap B^{c} \in \mathcal{A}$.

a) (8) Prove that $\mathcal{A}$ is a field.

b) (12) Let $\Omega = \{a, b, c, d\}$ and let $\mathcal{B} = \{B \subseteq \Omega : \#(B) \text{ is even}\}$, the sets with 0, 2, or 4 elements. Show that $\mathcal{B}$ is a $\lambda$-system. Is it a field?  
  ○ Yes  ○ No  Why?
Problem 2: For $0 < p < 1$ let $\{X_i : i \in \mathbb{N}\} \overset{\text{iid}}{\sim} \text{Ge}(p)$ be iid with the geometric probability distribution with probability mass function (pmf)

$$P[X_i = k] = pq^k, \quad k \in \mathbb{N}_0 \equiv \{0, 1, 2, \cdots \}, \quad q \equiv (1 - p).$$

a) (8) Find$^1$ the pmf for $Y_n \equiv \max_{1 \leq i \leq n} X_i$:

b) (8) Find$^2$ the pmf for $Z_n \equiv \min_{1 \leq i \leq n} X_i$:

c) (4) Find the chf (Characteristic Function) for $S_n \equiv \sum_{i \leq n} X_i$:

$^1$Suggestion: Find the CDFs for $X_i$ and then for $Y_n$ first.

$^2$What’s the probability that $Z_n$ is greater than $z$?
Problem 3: The random variables \( \{X_i\} \) are all independent and all satisfy \( \mathbb{E}[X_i^4] \leq 1.0 \), but they may have different distributions. Let \( S_n \equiv \sum_{i=1}^{n} X_i \) be their partial sum.

a) (8) Does it follow without any further assumptions that \( S_n / n \) converges almost surely?  
   ○ Yes  ○ No  Give a proof or counter-example.

b) (8) If in addition we know \( X_n \to 0 \) in probability, for which (if any) \( 0 < p < \infty \) does it follow that \( X_n \to 0 \) in \( L_p \)? Why?

c) (4) Give the best bound you can: (+1xc for showing it’s best possible)

\[
P[X_1 \geq 2] \leq \boxed{\text{______________}}
\]
Problem 4: Let $\Omega = \mathbb{R}_+ = [0, \infty)$ be the positive half-line, with Borel sets $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$ and probability measure $\mathbb{P}$ given by $\mathbb{P}(d\omega) = e^{-\omega} d\omega$ or, equivalently,

$$\mathbb{P}((a, b]] = e^{-a} - e^{-b} \quad 0 \leq a \leq b < \infty.$$ 

For each integer $n \in \mathbb{N} = \{1, 2, \cdots\}$ define a random variable on $(\Omega, \mathcal{F})$ by

$$X_n(\omega) := \begin{cases} 
0 & \text{if } \omega < n \\
1 & \text{if } \omega \geq n 
\end{cases}$$

a) (4) Find the mean $m_n = \mathbb{E}[X_n]$ for each $n \in \mathbb{N}$ and the covariance $\Sigma_{mn} = \mathbb{E}[(X_m - m_m)(X_n - m_n)]$ for each $m \leq n \in \mathbb{N}$:

$$m_n = \quad \Sigma_{mn} =$$

b) (4) Give the probability distribution measure $\mu_n(\cdot)$ of $X_n$ for each $n$: 

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Problem 4 (cont’d): As before, $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, $P(d\omega) = e^{-\omega} d\omega$, and $X_n(\omega) := 1_{[\infty, \infty)}(\omega)$ for $n \in \mathbb{N}$ (see footnote$^3$)

c) (4) For each fixed $n \in \mathbb{N}$ give the $\sigma$-algebra $\sigma(X_n)$ explicitly:

$$\sigma(X_n) = \left\{ \right\}$$

d) (4) Does the $\sigma$-algebra $\mathcal{G} = \sigma(X_1, X_2, \ldots)$ generated by all the $X_n$’s contain all the Borel sets in $\mathbb{R}_+$? ○ Yes ○ No If so, say why; if not, find a Borel set $B \in \mathcal{F}$ that is not in $\mathcal{G}$.

e) (4) Are $X_1$ and $X_2$ independent? ○ Yes ○ No Justify your answer.

$^3$Recall that the indicator random variable $1_A(\omega)$ is one if $\omega \in A$, otherwise zero.
Problem 5: As in Problem 4, $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, $P(d\omega) = e^{-\omega} d\omega$, and $X_n(\omega) := 1_{[n,\infty)}(\omega)$ for $n \in \mathbb{N}$.

a) (4) Prove that the partial sums $S_n := \sum_{j=1}^{n} X_j$ converge almost surely as $n \to \infty$ to some limiting random variable $S \equiv \sum_{j=1}^{\infty} X_j$.

b) (4) Do the partial sums $S_n := X_1 + \cdots + X_n$ converge to $S$ in $L_1$ as $n \to \infty$?  ○ Yes  ○ No  Justify your answer.

c) (4) Give the name\(^4\) and the mean of the probability distribution of the limit $S = \sum_{j=1}^{\infty} X_j$.

d) (8) Set $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}$, the $\sigma$-algebra generated by the first $n$ of the $X_k$’s. Find the indicated conditional expectations:

$E[X_1 \mid \mathcal{F}_2] =$  \hspace{1cm} $E[S \mid \mathcal{F}_2] =$

\(^4\)Remember, there’s a list of distributions with names and means at the back of this exam. Exactly what must $\omega$ be to make $S = 0$?  $S = 2$?  $S = k$?
Problem 6: Be specific for each of the following, leaving no parameters unspecified, but no need to prove convergence. For each part you may specify either the distributions themselves $\mu_n(dx)$, $\mu(dx)$ or random variables $X_n$, $X$ with those distributions.

a) (8) Give an example of a sequence of discrete distributions that converge in distribution to an absolutely-continuous distribution.

b) (8) Give an example of a sequence of absolutely-continuous distributions that converge in distribution to a discrete distribution.

c) (4) Give an example of a distribution supported on only rational values (so $\mu_n(B) = 1$ for some closed set $B \subset \mathbb{Q}$) that converges to one supported on only irrational values (so $\mu(B) = 1$ for some closed $B \subset \mathbb{Q}^c$).
Problem 7: Let \( \{X_j\}_{1 \leq j \leq 3} \) be independent random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) representing the outcomes on three independent fair 6-sided dice.

a) (6) How many points must \( \Omega \) have, at minimum? Why?

b) (6) Is it possible to find iid \( X_1, X_2, X_3 \) each uniform on \{1, 2, 3, 4, 5, 6\} on the space \((\Omega, \mathcal{F}, \mathbb{P})\) with \( \Omega = (0,1] \) and \( \mathbb{P} = d\omega \) Lebesgue measure on the Borel sets \( \mathcal{F} = \mathcal{B} \)? ◯ Yes ◯ No If so, give a possible version of \( X_1 : \Omega \to \mathbb{R} \) (+1xc for all three, \( X_1, X_2, X_3 \)); if not, why?

c) (8) Let \( Y = X_1 + X_2 \) and \( Z = X_2 + X_3 \). Find\(^5\): \( \mathbb{E}[Y \mid Z] = \)

\(^5\) Suggestion: First find \( \mathbb{E}[X_1 \mid X_2, X_3] \) and \( \mathbb{E}[X_2 \mid Z = X_2 + X_3] \).
Problem 8: Let $X_j \sim \text{Po}(1)$ be independent random variables, all with the unit-mean Poisson distribution.

a) (8) Find the characteristic function $\phi(\omega) = \mathbb{E}[e^{i\omega X_j}]$ of $X_j$ and the log chf $\psi(\omega) \equiv \log \phi(\omega)$.

b) (6) For numbers $a > 0$, find the log characteristic function $\psi_1(\omega)$ of $(X_j - 1)/a$.

c) (6) Let $S_n = X_1 + \cdots + X_n$ be the partial sum. Find a sequence $a_n > 0$ such that the log characteristic function $\psi_n(\omega)$ of $(S_n - n)/a_n$ converges to $-\omega^2/2$ for every $\omega$, and explain what this says about the limiting probability distribution of $S_n$ (i.e., about the Po($n$) distribution for large $n$).\footnote{Recall the Taylor series $e^x = 1 + x + x^2/2 + o(x^2) \approx 1 + x + x^2/2$ near $x \approx 0$.}
Problem 9: The random variables $X$ and $Y$ have a distribution generated by the following mechanism. A fair coin is tossed; if it falls Heads, then $X = Y = 0$; if it falls Tails, then $X$ and $Y$ are drawn independently from the standard normal $\text{No}(0, 1)$ distribution with CDF $\Phi(z) = \int_{-\infty}^{z} e^{-t^2/2} dt / \sqrt{2\pi}$.

a) (4) Are $X$ and $Y$ independent?  
   ○ Yes  ○ No  Why?

b) (4) Set $Z \equiv 3X + 4Y$. If the coin falls Tails (in which case $X, Y \overset{\text{iid}}{\sim} \text{No}(0, 1)$), find the conditional CDF for $Z$ (you may use $\Phi(\cdot)$ in your expression):

$$P[Z \leq z \mid \text{Tails}] =$$

c) (6) Now find the unconditional CDF for $Z = 3X + 4Y$:

$$P[Z \leq z] = \left\{ \right.$$  

and sketch a very very rough plot of it:
Problem 9 (cont’d):

d) (6) Let $\mathcal{G}$ be the $\sigma$-algebra generated by $Z$. Find the conditional expectation of $X$, given $\mathcal{G} = \sigma(Z)$:

$$E[X \mid \mathcal{G}] = \rule{2in}{0.1em}$$
Problem 10:  Let \( \{X_n > 0\} \) and \( X > 0 \) be positive random variables with \( X_n \to X \) a.s. Choose True or False below; no need to explain (unless you can’t resist). Each is 3pt, except g) (just 2pt, and difficult to get wrong).

a)  \( \begin{align*}
\text{\textbf{T} \ F} & \quad \log(X_n) \to \log(X) \text{ in probability.}
\end{align*} \)

b)  \( \begin{align*}
\text{\textbf{T} \ F} & \quad X_n \to X \text{ in } L_2 \text{ if each } E[|X_n|^3] \leq \pi.
\end{align*} \)

c)  \( \begin{align*}
\text{\textbf{T} \ F} & \quad \log(X_n) \to \log(X) \text{ in } L_1 \text{ if each } E[|X_n|^3] \leq \pi.
\end{align*} \)

d)  \( \begin{align*}
\text{\textbf{T} \ F} & \quad \limsup_{n \to \infty} E[\log(1 + X_n)] \geq E[\log(1 + X)].
\end{align*} \)

e)  \( \begin{align*}
\text{\textbf{T} \ F} & \quad X \in L_2 \text{ if, for some } t > 0, E[\exp(t \cdot X)] < \infty.
\end{align*} \)

f)  \( \begin{align*}
\text{\textbf{T} \ F} & \quad X \in L_2 \text{ if, for some } t < 0, E[\exp(t \cdot X)] < \infty.
\end{align*} \)

g)  \( \begin{align*}
\text{\textbf{T} \ F} & \quad \text{The answer you chose for this question is T.}
\end{align*} \)
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Another Blank Worksheet
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<th>Name</th>
<th>Notation</th>
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<th>Variance $\sigma^2$</th>
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<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0,1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$</td>
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<tr>
<td>Binomial</td>
<td>$\text{Bi}(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x q^{n-x}$</td>
<td>$x \in 0, \ldots, n$</td>
<td>$np$</td>
<td>$npq$</td>
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<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1 / \lambda$</td>
<td>$1 / \lambda^2$</td>
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<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha / \lambda$</td>
<td>$\alpha / \lambda^2$</td>
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<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = p q^{x}$</td>
<td>$x \in \mathbb{N}_0$</td>
<td>$q / p$</td>
<td>$q / p^2$</td>
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<td>HyperGeo.</td>
<td>$\text{HG}(n, A, B)$</td>
<td>$f(y) = \frac{\binom{n}{y} \binom{A+y}{n-y}}{\binom{A+B}{y}}$</td>
<td>$y \in {1, \ldots}$</td>
<td>$1 / p$</td>
<td>$q / p^2$</td>
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<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = \frac{e^{-(x-\mu)\beta}}{\sqrt{2\pi} \beta^\frac{3}{2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2 / 3$</td>
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<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{x \sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu + \sigma^2 / 2}$</td>
<td>$e^{2\mu + \sigma^2} \left( e^{\sigma^2} - 1 \right)$</td>
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<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, p)$</td>
<td>$f(x) = \binom{x+\alpha-1}{x} p^\alpha q^x$</td>
<td>$x \in \mathbb{N}_0$</td>
<td>$\alpha q / p$</td>
<td>$\alpha q / p^2$</td>
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<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
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<td>Pareto</td>
<td>$\text{Pa}(\alpha, \epsilon)$</td>
<td>$f(x) = \frac{\alpha \epsilon^\alpha}{x^{\alpha+1}}$</td>
<td>$x \in (\epsilon, \infty)$</td>
<td>$\frac{\epsilon \alpha}{\alpha-1}$</td>
<td>$\frac{\epsilon^\alpha}{(\alpha-1)^2 (\alpha-2)}$</td>
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<tr>
<td>Poisson</td>
<td>$\text{Po}(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$x \in \mathbb{N}_0$</td>
<td>$\lambda$</td>
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<td>Snedecor $F$</td>
<td>$\text{F}(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \times x^{-\frac{\nu_1}{2}} \left[1 + \frac{\nu_1}{\nu_2} x\right]^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_2}{\nu_2-2}$</td>
<td>$\left(\frac{\nu_2}{\nu_2-2}\right)^{\frac{\nu_1}{2} / \nu_2} \left(1 + \frac{\nu_1}{\nu_2} x\right)^{\frac{\nu_1}{2} / \nu_2}$</td>
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<tr>
<td>Student $t$</td>
<td>$\text{t}(\nu)$</td>
<td>$f(x) = \frac{\nu^{\frac{\nu+1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi}} \left[1 + x^2 / \nu\right]^{-(\nu+1)/2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$</td>
<td>$\nu / (\nu - 2)$</td>
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<td>Uniform</td>
<td>$\text{Un}(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
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<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \lambda)$</td>
<td>$f(x) = \lambda \alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\Gamma(1+\alpha^{-1})}{\lambda}$</td>
<td>$\frac{\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)}{\lambda^2}$</td>
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