Midterm Examination I

STA 711: Probability & Measure Theory

Thursday, 2012 Oct 11, 11:45 am – 1:00pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing please ask me—don’t guess, and don’t discuss questions with others.

Unless a problem states otherwise, you must show your work. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.
**Problem 1.** Let $\Omega = \{a, b, c, d\}$ have just four points, with $\sigma$-algebra $\mathcal{F} = 2^\Omega$ and probability assignment $\mathbb{P}[A] = \sum_{i=1}^{4} \frac{i}{10} \mathbf{1}_A(\omega_i)$ to events $A \in \mathcal{F}$, where $\omega_1 = a$, $\omega_2 = b$, $\omega_3 = c$ and $\omega_4 = d$. Define a collection of sets by

$$\mathcal{G} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, \Omega\}$$

a) (8) Give explicitly a real random variable $X$ that generates $\mathcal{G} = \sigma(X)$:

$$X(a) = \underline{\quad} \quad X(b) = \underline{\quad} \quad X(c) = \underline{\quad} \quad X(d) = \underline{\quad}$$

b) (6) Give explicitly a real random variable $Y$ that takes only two distinct values, for which $\mathcal{F} = \sigma(X, Y)$:

$$Y(a) = \underline{\quad} \quad Y(b) = \underline{\quad} \quad Y(c) = \underline{\quad} \quad Y(d) = \underline{\quad}$$

c) (6) Find the expectation of your random variables $X$ and $Y$ above:

$$E[X] = \underline{\quad} \quad E[Y] = \underline{\quad}$$
Problem 2. Again let \( \Omega = \{a, b, c, d\} \) with \( \mathcal{F} = 2^\Omega \) and \( P \) that assigns probabilities 1/10, 2/10, and 3/10 respectively to the singleton sets \{a\}, \{b\} and \{c\}. Consider the two fields

\[
\mathcal{C}_1 = \{\emptyset, \{a, b\}, \{c, d\}, \Omega\}
\]
\[
\mathcal{C}_2 = \{\emptyset, \{a, c\}, \{b, d\}, \Omega\}
\]

a) (8) Are \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) independent? Give a proof or counterexample.

Y N  Why?

b) (6) Find a real random variable \( X \) that is \( \mathcal{C}_2 \setminus \mathcal{B} \)-measurable but not \( \mathcal{C}_1 \setminus \mathcal{B} \)-measurable (be careful not to mix up 1 and 2).

\[
X(a) = \quad X(b) = \quad X(c) = \quad X(d) =
\]

c) (6) Find all random variables that are both \( \mathcal{C}_2 \setminus \mathcal{B} \) and \( \mathcal{C}_1 \setminus \mathcal{B} \)-measurable. Justify your answer.
Problem 3. Let \( \{U_n\} \) be independent random variables with uniform distributions on \((0,1]\) and let \( \{p_n\} \) be (non-random) numbers in \((0,1)\). Set:

\[
X_n = 1_{\{U_n \leq p_n\}} \quad Y_N = \prod_{1 \leq n \leq N} X_n,
\]

each taking the values 0 or 1.

a) (8) What is the probability distribution of \( X_n \)?

\[
\mu_{X_n}(B) = \]

b) (6) If possible, find \[\text{and box}\] a sequence \( \{p_n\} \) for which you can show the event

\[
\left[ \lim_{N \to \infty} Y_N > 0 \right]
\]

has positive probability; if this is not possible, explain why.

c) (6) If possible, find \[\text{and box}\] a sequence \( \{p_n\} \) for which you can show the event

\[
\left[ \sum_{N=1}^{\infty} X_N < \infty \right]
\]

has positive probability; if this is not possible, explain why.
Problem 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the natural numbers $\Omega = \mathbb{N} = \{1, 2, 3, \ldots \}$ with $\mathcal{F} = 2^\Omega$ and $\mathbb{P}[A] = \sum \{2^{-\omega} : \omega \in A \cap \mathbb{N} \}$.

a) (7) Fix $\lambda \in \mathbb{R}$. Is the random variable $X(\omega) = e^{\lambda \omega}$ in $L_1(\Omega, \mathcal{F}, \mathbb{P})$? If so, find $\mathbb{E}[X]$ in closed form; if not, tell why; if this depends on $\lambda$, explain.
○ Yes  ○ No  ○ It Depends  Reasoning:

b) (7) For $n \in \mathbb{N}$ define a random variable $Y_n$ by $Y_n(\omega) = n$ if $\omega \geq n$, $Y_n(\omega) = 0$ if $\omega < n$. Does the Dominated Convergence Theorem apply to $\{Y_n\}$? If so, tell what DCT says and show why it applies; if not, explain why.
○ Yes  ○ No  Reasoning:

c) (6) Define $Y_n$ as above. Does Fatou’s Lemma apply? If so, verify Fatou’s conclusion by calculation; if not, why?  ○ Yes  ○ No  Reasoning:
Problem 5. Let \( X = \omega (2 - \omega) \) be a random variable on the space \( \Omega = (0, 2] \) with \( \mathcal{F} = \mathcal{B}(\Omega) \), the Borel sets (it’s plotted below).

a) (7) Find and plot a non-negative simple random variable \( Y \in \mathcal{E}_+ \) satisfying \( 0 \leq Y(\omega) \leq X(\omega) \) and \( |X(\omega) - Y(\omega)| \leq 0.4 \) for all \( \omega \in \Omega \).

\[
\begin{align*}
Y(\omega) =
\end{align*}
\]

b) (7) Find \( EX \) and \( EY \) for the probability measure \( P(d\omega) = d\omega/2 \) (i.e.,
\[
P \{ (a, b] \} = (b - a)/2 \text{ for all } 0 \leq a \leq b \leq 2:
\]

\[
\begin{align*}
EX = & \quad EY =
\end{align*}
\]

c) (6) Let \( Z = 1_{[0,1]}(\omega) \). Are \( X \) and \( Z \) independent on \( (\Omega, \mathcal{F}, P) \)?

\[
\begin{array}{ll}
\text{Yes} & \text{No} \quad \text{Why?}
\end{array}
\]
Blank Worksheet
Another Blank Worksheet