Midterm Examination II

STA 711: Probability & Measure Theory

Thursday, 2012 Nov 15, 11:45 am – 1:00pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing, *please* ask me to clarify it.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets and a pdf/pmf sheet at the end of the test. It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find. Good luck.

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Print Name: ______________________________

Problem 1. Let $\Omega = \mathbb{R}_+$ with Borel sets $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$ and probability measure

$$\mathsf{P}[A] = \int_A e^{-\omega}\, d\omega$$

for $A \in \mathcal{F}$. For $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ set $X_n(\omega) \equiv \omega^n$.

a) (6) Find (explicitly, in closed form— *simplify*) the bound that a direct application of Markov's inequality[1] gives for
$\mathsf{P}[X_3 \ge 8] \le$

b) (6) Find (in closed form— *simplify*) the exact probability
$\mathsf{P}[X_3 \ge 8] =$

c) (8) Set $Z \equiv \sum_{0 \le n < \infty} X_n/n!$. Evaluate $Z(\omega)$ explicitly and find $\mathsf{E}[Z^p]$ for each $p > 0$:

$Z(\omega) =$ ______________ $\mathsf{E}[Z^p] =$ ______________

[1] You can compute $\mathsf{E}X_n$ explicitly, using the Gamma or factorial functions.

Problem 2: Let $\{X_i, Y_i : \ i \in \mathbb{N}\}$ be iid with the standard exponential $\mathsf{Ex}(1)$ distribution[2]. Set $Z_i \equiv (X_i - Y_i)$ and $S_n \equiv \sum_{1\le i\le n} Z_i$.

a) (5) Find the characteristic function for S_n:[3]
$\phi_n(\omega) \equiv \mathsf{E}e^{i\omega S_n} =$

b) (5) Find the mean and variance of S_n by any method you wish (but show your work or explain your answer):

$\mu_n \equiv \mathsf{E}S_n =$ ______________ $\sigma_n^2 \equiv \mathsf{E}(S_n - \mu_n)^2 =$ ______________

[2] A sheet of information about common distributions is at the back of this exam.
[3] Suggestion: First find the ch.f. for X_1; then for $-Y_1$; then for Z_1; then for S_n.

Problem 2 (cont'd):

c) (5) Find the indicated limits for $\omega \in \mathbb{R}$ as $n \to \infty$:

$\phi_n(\omega/n) \to$ ______________ $\qquad S_n/n \Rightarrow$ ______________

d) (5) Find the indicated limits for $\omega \in \mathbb{R}$ as $n \to \infty$:

$\phi_n(\omega/\sqrt{n}) \to$ ______________ $\qquad S_n/\sqrt{n} \Rightarrow$ ______________

Problem 3: Let $\{X_n\}$ be independent real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ for $n \in \mathbb{N}$, with a common *continuous* distribution. Call X_n a "record" if $X_n > \max\{X_j : 1 \le j < n\}$ (X_1 is always a record), and set:

$$\zeta_n \equiv \begin{cases} 1 & X_n \text{ is a record} \\ 0 & X_n \text{ is not a record.} \end{cases}$$

a) (5) Find[4]

$\mathsf{E}[\zeta_n] = \mathsf{P}[X_n \text{ is a record}] =$ ________

b) (5) Are $\{\zeta_2, \zeta_3\}$ independent? ◯ Yes ◯ No Why?[4]

[4]For parts a) b) c) of this problem it is only the *order* of the $\{X_n\}$ that matter, not their specific values. What *are* the possible orders of, say, X_1, X_2, X_3? What are their probabilities? Recall that they are iid with a continuous distribution. Symmetry helps.

Problem 3 (cont'd):

c) (5) Let $Z_n \equiv \sum_{j=1}^n \zeta_j$ be the number of records among the first n observations. Prove $Z_n/n \to 0$ in L_1 (for 4pts) and *a.s.* (for 1 pt).

d) (5) Which of the preceding answers in this Problem 3 would change if the common distribution of $\{X_n\}$ were not continuous? Give an example to illustrate.[5] Circle the ones that *would* change: a) b) c) and explain:

[5] Suggestion: consider iid Bernoulli $\{X_n\}$ with $p = 1/2$.

Problem 4: Let X be a discrete-valued random variable with pmf $p_n = \mathsf{P}[X = a_n]$ for some $\{a_n\} \subset \mathbb{R}$, $\{p_n\} \subset \mathbb{R}_+$ s.t. $\sum_{n=1}^{\infty} p_n = 1$ and let Y be an absolutely-continuous real valued random variable with pdf $g(y)$.

a) (4) Exactly what does it *mean* for X and Y to be independent? Give either the definition or a sufficient criterion.

b) (4) Give an expression (it should involve $\{a_n\}$, $\{p_n\}$, and g) for the indicated expectation, for a bounded measurable $h: \mathbb{R}^2 \to \mathbb{R}$:
$\mathsf{E}\big[h(X,Y)\big] =$

c) (6) If X and Y are independent, is their sum $Z \equiv X + Y$ ◯ *discrete*, ◯ *absolutely continuous*, or ◯ *can't tell*?? If discrete, give the pmf $p(z)$; if absolutely continuous, give the pdf $f(z)$; if this can't be determined, explain.

d) (6) Give the exact conditions on $\{a_n\}$, $\{p_n\}$, and g needed to ensure that $X \in L_2$ and $Y \in L_2$.

Problem 5: True or false? Circle one; each answer is 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky (no tricks are intended). All random variables are real.

a) T F Lebesgue's dominated convergence theorem implies that $\int_0^1 \sin(nx)\,dx \to 0$ as $n \to \infty$.

b) T F Jensen's Inequality implies that $\mathsf{E}(X^2) \geq (\mathsf{E}X)^2$ for $X \in L_1$.

c) T F For any r.v. Y and number $a > 0$, $\mathsf{P}[Y > a] \leq \mathsf{E}[Y^2]/a^2$.

d) T F For any sequence of random variables $\{X_n\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$,

$$\mathsf{E}\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} [\mathsf{E}X_n].$$

e) T F For any random variables $\{X_\alpha\}$, $\{\cos(X_\alpha)\}$ is UI.

f) T F Let X have the geometric distribution with $\mathsf{P}[X = k] = 2^{-k-1}$ for $k \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. Then $\mathsf{P}[X \text{ is odd}] \geq 1/2$.

g) T F Three σ-fields $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ are independent if and only if $\mathsf{P}[F_i \cap F_j] = \mathsf{P}[F_i]\ \mathsf{P}[F_j]$ for every $F_i \in \mathcal{F}_i$, for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

h) T F Random variables X and Y are independent if and only if $\mathsf{E}[f(X) \cdot g(Y)] = \mathsf{E}[f(X)] \cdot \mathsf{E}[g(Y)]$ for all bounded Borel functions $f(x)$, $g(y)$.

i) T F If $\{X_n\}$ is UI then for some constant $B > 0$ each $\|X_n\|_1 \leq B$.

j) T F If X_n is absolutely continuous with pdf $f_n(x)$ and if X_n converges in distribution, then $f_n(x)$ converges as $n \to \infty$ to the pdf $f(x)$ of the limiting distribution, for every x where $f(x)$ is continuous.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\, e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{N}_0$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{N}_0$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{N}_0$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\lambda)$	$f(x)=\alpha\lambda^\alpha\,x^{\alpha-1}\,e^{-(\lambda x)^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\lambda}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\lambda^2}$	