# Sta 711 : Homework 1

## Fields and $\sigma$ - fields

1. Enumerate the class $\aleph$ of all $\sigma$-fields $\mathcal{F}$ on the three-point set $\Omega = \{a, b, c\}$ that contain the singleton $\{a\}$, *i.e.*, that satisfy $\mathcal{C} \subset \mathcal{F}$ for $\mathcal{C} := \big\{\{a\}\big\}$. Also find $\sigma(\mathcal{C})$.

2. For each integer $n \in \mathbb{N} := \{1, 2, 3, ...\}$, set
$$A_n := \left\{\frac{m}{n} : \quad m \in \mathbb{N}\right\}.$$
Find $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

3. Let $\{f_n\}$ be real-valued functions on any set $\Omega$. The sequence *converges* at a point $\omega \in \Omega$ if and only if for **every one** of the uncountably-many numbers $\epsilon > 0$ there exists some $m \in \mathbb{N}$ such that $|f_n(\omega) - f_m(\omega)| < \epsilon$ for all $n \geq m$. Let $\epsilon_k > 0$ be any fixed sequence such that $\epsilon_k \searrow 0$ as $k \to \infty$. Show that:
$$\{\omega : \ f_n(\omega) \text{ converges}\} = \cap_{k=1}^{\infty} \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} \{\omega : \ |f_n(\omega) - f_m(\omega)| < \epsilon_k\}.$$

4. Find a set $\Omega$ and two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\Omega$ for which $\mathcal{F}_1 \cup \mathcal{F}_2$ is *not* a field.

5. Suppose a collection $\{\mathcal{F}_n : \ n \in \mathbb{N}\}$ of $\sigma$-fields satisfies the relation $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for every $j \in \mathbb{N}$. Does it follow that $\cup\mathcal{F}_j$ is a field? (the answer is "yes"— show why)

6. Under the same conditions, is $\cup\mathcal{F}_j$ is a $\sigma$-field? (this one is "no"— find a counter-example. The idea is to find a sequence $A_n \in \mathcal{F}_n$ with $\cup_n A_n \notin \mathcal{F}_j$ for every $j$, hence $\cup_n A_n \notin \cup_j \mathcal{F}_j$).

**Dyadic Rationals**

For problems 7–10, let $\Omega = \mathbb{Q} := \left\{ j/2^n : \ j \in \{1, 2, \cdots, 2^n\}, \ n \in \mathbb{N} \right\}$ be the dyadic rational numbers in the half-open unit interval, and let $\mathcal{C} = \{(0, b] : \ b \in \mathbb{Q}, \ 0 < b \leq 1\}$ denote the collection of half-open intervals $(0, b] = \{q \in \mathbb{Q} : \ 0 < q \leq b\}$ with left endpoint zero.

Recall that a real-valued set function $\mathsf{P}$ on a $\sigma$-algebra $\mathcal{G}$ of subsets of a space $\Omega$ is a "probability measure" if and only if it satisfies the three rules:

- $(\forall A \in \mathcal{G})$ $\mathsf{P}(A) \geq 0$;
- $(\forall \{A_i\} \subset \mathcal{G}, \ A_i \cap A_j = \emptyset)$ $\mathsf{P}(\cup A_i) = \sum \mathsf{P}(A_i)$;
- $\mathsf{P}(\Omega) = 1$.

7. Let $n \in \mathbb{N}$ be a FIXED positive integer (like three) and set

$$\mathcal{B}_n := \{(0, j/2^n], \ j \in \{0, 1, ..., 2^n\}\},$$

the collection of half-open intervals of dyadic rationals from zero up to an integral multiple of $2^{-n}$. Describe the elements of the $\sigma$-field

$$\mathcal{F}_n := \sigma(\mathcal{B}_n)$$

generated by $\mathcal{B}_n$. How many elements does $\mathcal{B}_n$ have? How many distinct elements does $\mathcal{F}_n$ have? What are they? Suggestion: Try $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ first, by hand.

8. Describe explicitly[1] the field $\mathcal{F}_0 := \mathcal{F}(\mathcal{C})$ of subsets of $\mathbb{Q}$ generated by $\mathcal{C}$ (hint: Do problems (5) and (7) first).

9. Describe explicitly[1] the $\sigma$-field $\mathcal{F} := \sigma(\mathcal{C})$ of subsets of $\mathbb{Q}$ generated by $\mathcal{C}$.

10. Define a set function $\lambda_0$ on $\mathcal{C}$ by

$$\lambda_0\big( \ (0, b] \ \big) = b$$

Show that there does **not** exist a probability measure $\lambda$ on $(\mathbb{Q}, \mathcal{F})$ that extends $\lambda_0$, *i.e.*, one for which $\lambda\big( \ (0, b] \ \big) = b$ for all $b \in \mathbb{Q}$.

[1] Here's an example to illustrate what makes a description "explicit". An *explicit* description of $\sigma(\{\omega\} : \ \omega \in \Omega)$, is: "All finite or co-finite subsets of $\Omega$." Just recounting the definition, "the smallest $\sigma$-algebra that contains each singleton $\{\omega\} \subset \Omega$," is not explicit in this sense.