Sta 711 : Homework 2

σ-Algebras and partitions.

Fields and σ-fields generated by *partitions* (finite or countable collections of *disjoint* events $\Lambda_j \in \mathcal{F}$ with $\cup \Lambda_j = \Omega$), and probability assignments on them, area especially easy to describe.

1. Let $\{A, B, C\} \subset \mathcal{F}$ be three events in a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, not necessarily non-empty or disjoint. Enumerate all possible elements of the partition $\mathcal{P} = \mathcal{P}(A, B, C)$ generated by these events (*i.e.*, the smallest partition for which $\{A, B, C\} \subset \sigma(\mathcal{P})$). How many distinct nonempty elements does $\mathcal{P}$ have, at most? How many, at minimum?

2. How many distinct elements does the σ-algebra $\sigma(\mathcal{P})$ contain, at most? At minimum? Describe them in words (don't list them).

3. Let's further assume that the events A, B, C are disjoint and $\Omega = A \cup B \cup C$, with probabilities $\mathsf{P}(A) = 0.6$ and $\mathsf{P}(B) = 0.3$. Calculate the probability of every event in $\sigma(A, B, C)$.

Null sets.

4. Let $\{A_n, n \in \mathbb{N}\}$ be events with $\mathsf{P}(A_n) = 1$. Prove that $\mathsf{P}(\cap_{n=1}^{\infty} A_n) = 1$.

5. Now consider uncountably many events $\{B_\alpha\}$, all with $\mathsf{P}(B_\alpha) = 0$. Does it follow necessarily that $\mathsf{P}(\cup_\alpha B_\alpha) = 0$? Give a proof or a counter example.

6. Let $\{C_k\}$ be a collection of events such that $\sum_{k=1}^{n} \mathsf{P}(C_k) > n-1$ for some $n \in \mathbb{N}$. Show that $\mathsf{P}(\cap_{k=1}^{n} C_k) > 0$.

Distribution functions and continuity.

7. Give an example of a real-valued function on $\mathbb{R}$ which in continuous, but **not** uniformly continuous.

8. Let G be a continuous distribution function on $\mathbb{R}$. Show that G is in fact uniformly continuous. Hint: Consider points $\{x_i\}$ for which $G(x_i) = i/n$ for $1 \le i < n$. Are these $\{x_i\}$ determined uniquely? Does that matter?

9. Show that any distribution function F on $\mathbb{R}$ can have *at most countably many* discontinuities. Hint: Consider the open intervals $(F(x-), F(x))$ for discontinuity points x.

π- & λ- systems.

10. Consider the following collecton of subsets of the real line:

$$\mathcal{B} = \{(-\infty, b] : \ b \in \mathbb{R}\}$$

   (a) Show that $\mathcal{B}$ is a π- system, but not a λ system.

   (b) What is the λ- system generated by $\mathcal{B}$? Why?

11. Consider the following collections of subsets of the unit square $\Omega = (0,1]^2 \subset \mathbb{R}^2$:

$$\mathcal{A} = \{(a,b] \times (c,d] : \ 0 \le a \le b \le 1, \ 0 \le c \le d \le 1\}$$

   (a) Is $\mathcal{A}$ a π- system? Why or why not?

   (b) Is $\mathcal{A}$ a λ- system? Why or why not?

π - systems and fields.

Let $\mathcal{C}$ be a non-empty collection of subsets of a space Ω.

12. Let $\mathcal{F}(\mathcal{C})$ be the smallest field containing $\mathcal{C}$. Show that for each $B \in \mathcal{F}(\mathcal{C})$ there exists a *finite* subcollection $\mathcal{C}' \subseteq \mathcal{C}$ for which $B \in \mathcal{F}(\mathcal{C}')$. Note $\mathcal{C}'$ may depend on B.

13. Show that the smallest field containing $\mathcal{C}$ consists precisely of sets of the form

$$\mathcal{F}(\mathcal{C}) = \big\{B : \ B = \cup_{i=1}^m B_i, \quad B_i = \cap_{j=1}^{n_i} A_{ij} \text{ for some } m \in \mathbb{N}, \ \{n_i\} \subset \mathbb{N}\big\}$$

where for each index pair (i,j), either $A_{ij} \in \mathcal{C}$ or $A_{ij}^c \in \mathcal{C}$, and where the m sets $\{B_i\}$ are disjoint. Thus, we can represent the sets in $\mathcal{F}(\mathcal{C})$ explicitly (interestingly, it turns out to be impossible to do this for σ-fields).

14. Suppose that Show that if two probability measures P_1, P_2 agree on a π system $\mathcal{C}$, then they must also agree on the field $\mathcal{F}(\mathcal{C})$ generated by $\mathcal{C}$. Hint: Use Dynkin's π-λ theorem, or part (13) and the inclusion-exclusion principle.

15. Find two probability measures P_1, P_2 on some set Ω that agree on a collection of subsets $\mathcal{C}$, but *not* on $\mathcal{F}(\mathcal{C})$. Obviously from (14) above $\mathcal{C}$ cannot be a π-system. Hint: It's enough to have $\mathcal{C} = \{A, B\}$ with just two elements, on an outcome space Ω with just four points. Would three points be enough?