Sta 711: Homework 2

$\sigma$-Algebras and partitions.

Fields and $\sigma$-fields generated by partitions (finite or countable collections of disjoint events $\Lambda_j \in \mathcal{F}$ with $\cup \Lambda_j = \Omega$), and probability assignments on them, are especially easy to describe.

1. Let $\{A, B, C\} \subset \mathcal{F}$ be three events in a probability space $(\Omega, \mathcal{F}, P)$, not necessarily non-empty or disjoint. Enumerate all possible elements of the partition $\mathcal{P} = \mathcal{P}(A, B, C)$ generated by these events (i.e., the smallest partition for which $\{A, B, C\} \subset \sigma(\mathcal{P})$). How many distinct nonempty elements does $\mathcal{P}$ have, at most? How many, at minimum?

2. How many distinct elements does the $\sigma$-algebra $\sigma(\mathcal{P})$ contain, at most? At minimum? Describe them in words (don’t list them).

3. Let’s further assume that the events $A, B, C$ are disjoint and $\Omega = A \cup B \cup C$, with probabilities $P(A) = 0.6$ and $P(B) = 0.3$. Calculate the probability of every event in $\sigma(A, B, C)$.

Null sets.

4. Let $\{A_n, n \in \mathbb{N}\}$ be events with $P(A_n) = 1$. Prove that $P(\cap_{n=1}^{\infty} A_n) = 1$.

5. Now consider uncountably many events $\{B_\alpha\}$, all with $P(B_\alpha) = 0$. Does it follow necessarily that $P(\cup_\alpha B_\alpha) = 0$? Give a proof or a counter example.

6. Let $\{C_k\}$ be a collection of events such that $\sum_{k=1}^{n} P(C_k) > n - 1$ for some $n \in \mathbb{N}$. Show that $P(\cap_{k=1}^{n} C_k) > 0$.

Distribution functions and continuity.

7. Give an example of a real-valued function on $\mathbb{R}$ which is continuous, but not uniformly continuous.

8. Let $G$ be a continuous distribution function on $\mathbb{R}$. Show that $G$ is in fact uniformly continuous. Hint: Consider points $\{x_i\}$ for which $G(x_i) = i/n$ for $1 \leq i < n$. Are these $\{x_i\}$ determined uniquely? Does that matter?

9. Show that any distribution function $F$ on $\mathbb{R}$ can have at most countably many discontinuities. Hint: Consider the open intervals $(F(x^-), F(x))$ for discontinuity points $x$. 

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π- & λ- systems.

10. Consider the following collection of subsets of the real line:

\[ B = \{(-\infty, b] : b \in \mathbb{R}\} \]

(a) Show that \( B \) is a π-system, but not a λ system.

(b) What is the λ-system generated by \( B \)? Why?

11. Consider the following collections of subsets of the unit square \( \Omega = (0,1]^2 \subset \mathbb{R}^2 \): 

\[ \mathcal{A} = \{(a,b] \times (c,d] : 0 \leq a \leq b \leq 1, \ 0 \leq c \leq d \leq 1\} \]

(a) Is \( \mathcal{A} \) a π-system? Why or why not?

(b) Is \( \mathcal{A} \) a λ-system? Why or why not?

π - systems and fields.

Let \( \mathcal{C} \) be a non-empty collection of subsets of a space \( \Omega \).

12. Let \( \mathcal{F}(\mathcal{C}) \) be the smallest field containing \( \mathcal{C} \). Show that for each \( B \in \mathcal{F}(\mathcal{C}) \) there exists a finite subcollection \( \mathcal{C}' \subseteq \mathcal{C} \) for which \( B \in \mathcal{F}(\mathcal{C}') \). Note \( \mathcal{C}' \) may depend on \( B \).

13. Show that the smallest field containing \( \mathcal{C} \) consists precisely of sets of the form

\[ \mathcal{F}(\mathcal{C}) = \{B : B = \bigcup_{i=1}^{m} B_i, \ B_i = \bigcap_{j=1}^{n_i} A_{ij} \text{ for some } m \in \mathbb{N}, \ \{n_i\} \subset \mathbb{N}\} \]

where for each index pair \((i, j)\), either \( A_{ij} \in \mathcal{C} \) or \( A_{ij}^c \in \mathcal{C} \), and where the \( m \) sets \( \{B_i\} \) are disjoint. Thus, we can represent the sets in \( \mathcal{F}(\mathcal{C}) \) explicitly (interestingly, it turns out to be impossible to do this for σ-fields).

14. Suppose that Show that if two probability measures \( \mathbf{P}_1, \mathbf{P}_2 \) agree on a π system \( \mathcal{C} \), then they must also agree on the field \( \mathcal{F}(\mathcal{C}) \) generated by \( \mathcal{C} \). Hint: Use Dynkin’s π-λ theorem, or part (13) and the inclusion-exclusion principle.

15. Find two probability measures \( \mathbf{P}_1, \mathbf{P}_2 \) on some set \( \Omega \) that agree on a collection of subsets \( \mathcal{C} \), but not on \( \mathcal{F}(\mathcal{C}) \). Obviously from (14) above \( \mathcal{C} \) cannot be a π-system. Hint: It’s enough to have \( \mathcal{C} = \{A, B\} \) with just two elements, on an outcome space \( \Omega \) with just four points. Would three points be enough?