Sta 711: Homework #3

Random variables
1. Let $(\Omega, \mathcal{F}, P) = ((0,1], \mathcal{B}, \lambda)$ for Lebesgue measure $\lambda$ on the Borel sets of the unit interval. For $\omega \in \Omega$ define:

$$X_1(\omega) \equiv \min(\omega, 0.6) \quad X_2(\omega) \equiv 1_{(0,1/3]}(\omega) \quad X_3(\omega) \equiv \sqrt{\omega}$$

Plot each of the CDFs $F_k(x) \equiv P[X_k \leq x], x \in \mathbb{R}$, and describe explicitly the $\sigma$-algebras $\mathcal{F}_k := \sigma(X_k)$.

2. Let $X$ be a random variable with CDF $F(x) := P(X \leq x)$. Set $Y \equiv F(X)$. If $X$ has a continuous distribution (i.e., if $F$ is a continuous function), show that $Y$ is a random variable and that $Y$ has a uniform distribution on $[0,1]$. Warning: $F(x)$ may not be strictly increasing, and so may not be one-to-one.

3. A random variable $Y$ is bounded if there is a fixed finite number $0 \leq B < \infty$ for which $|Y(\omega)| \leq B$ for all $\omega \in \Omega$. Give an example of a real-valued random variable $X$ that each $X(\omega)$ is finite but $X$ is not bounded.

4. Let $X$ be a real valued random variable (so $P[|X| < \infty] = 1$) with CDF $F(x)$. For each $\epsilon > 0$, find (explicitly) a bounded random variable $Y_\epsilon$ such that

$$P(X \neq Y_\epsilon) < \epsilon.$$

Measurable functions
5. Let $\Omega = \mathbb{R}$. Show that $\mathcal{S} := \{\emptyset, (-\infty,0], (0,\infty), \Omega\}$ is a $\sigma$-algebra. Describe all functions $f : \Omega \to \mathbb{R}$ that are $\mathcal{S}\setminus\mathcal{B}$-measurable.

6. If $X$ is a real-valued random variable on any probability space $(\Omega, \mathcal{F}, P)$, then show that $|X|$ is also a random variable. Show by an example that the converse need not be true (Hint: A finite $\Omega$ will suffice)

7. Let $\Omega = \mathbb{R}$, and let $\mathcal{S}_0 \equiv \{\emptyset, \Omega\}$ be the trivial $\sigma$-algebra. Find all measurable functions $X : (\Omega, \mathcal{S}_0) \to (\mathbb{R}, \mathcal{B})$.

8. Let $\mathcal{F}_X := \sigma(X)$ be the $\sigma$-algebra generated by the function $X(\omega) = \omega^2$ on $\Omega = \mathbb{R}$. Is the set $A = (-\infty, 0]$ in $\mathcal{F}_X$? How about $B = [-4, 4]$? Why?

9. Let $\{X_n, n \geq 0\}$ be real-valued random variables on $(\Omega, \mathcal{F}, P)$ that satisfy

$$\limsup_{n \to \infty} X_n(\omega) = +\infty$$
for every $\omega \in \Omega$, and let $B < \infty$ be a real number. Prove that the integer-valued quantity 

$$\tau(\omega) \equiv \inf \{n \geq 0 : X_n(\omega) \geq B\}$$

is a random variable.

**Extra credit:** Prove that $X$ is also a random variable.

## Random Variables and $\sigma$-Algebras

All parts of this problem concern the same probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = (0, 1]$, $\mathcal{F} = B(\Omega)$ the Borel sets, and $P = \lambda$ Lebesgue measure. Let $\delta_n(\omega)$ be the $n$th bit in the binary expansion of $\omega$, given by

$$\delta_n(\omega) := [1 + 2^n \omega] \pmod{2}$$

where $[x]$ is the least integer $\geq x$, and set

$$\mathcal{F}_n := \sigma \{\delta_1, \ldots, \delta_n\} = \sigma \{(0, j/2^n] : j = 0, \ldots, 2^n\}.$$ 

(a) Find a single real-valued random variable $X$ on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_3 = \sigma(X)$.

(b) True or False: If $Y$ is any other random variable on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_3 = \sigma(Y)$, then $Y = g(X)$ for some Borel measurable function $g : \mathbb{R} \to \mathbb{R}$. Give a proof or counter-example.

(c) Let $Z$ be a random variable on $(\Omega, \mathcal{F}, P)$ for which $\mathcal{F} = \sigma(Z)$.

True or false: For each $\omega_1 \neq \omega_2$, necessarily $Z(\omega_1) \neq Z(\omega_2)$. Explain.