Sta 711 : Homework 5

1. Independence.
   (a) Let \( \{B_i\} \) be independent events. For \( n \in \mathbb{N} \) show that
   \[
   P \left( \bigcup_{i=1}^{n} B_i \right) = 1 - \prod_{i=1}^{n} [1 - P(B_i)] \geq 1 - \exp \left\{ - \sum_{i=1}^{n} P(B_i) \right\}
   \]
   (b) If \( \{A_n, n \in \mathbb{N}\} \) is a sequence of events such that
   \[
   P(A_n \cap A_m) = P(A_n)P(A_m) \quad \forall n, m \in \mathbb{N}, \ n \neq m,
   \]
   does it follow that the events \( \{A_n\} \) are independent? Give a proof or counterexample.
   (c) Let \( Y \) be a random variable. Show that \( Y \) is independent of itself if and only if, for some constant \( c \in \mathbb{R} \), \( P[Y = c] = 1 \).
   Let \( f : \mathbb{R} \to \mathbb{R} \) be Borel measurable, and \( X \) any random variable. Can \( f(X) \) and \( X \) be independent? Explain your answer.
   (d) Give an example to show that an event \( A \in \mathcal{F} \) may be independent of each \( B \)
   in some collection \( \mathcal{C} \subset \mathcal{F} \) of events, but not independent of \( \sigma(\mathcal{C}) \). Prove this is impossible if \( \mathcal{C} \) is a \( \pi \)-system (i.e., in that case \( A \) must be independent of \( \sigma(\mathcal{C}) \)).
   (e) Give a simple example to show that two random variables on the same space
   \((\Omega, \mathcal{F})\) may be independent according to one probability measure \( P_1 \) but dependent with respect to another \( P_2 \).

2. Borel Cantelli.
   (a) Let \( \{X_n\} \) be a sequence of Bernoulli random variables with
   \[
   P(X_n = 1) = n^{-p} \quad P(X_n = 0) = 1 - n^{-p}
   \]
   for some \( p > 0 \). For \( p = 2 \) show that the partial sum
   \[
   S_n := \sum_{k=1}^{n} X_k
   \]
   converges almost-surely, whether or not the \( \{X_n\} \) are independent. If the \( \{X_n\} \)
   are independent, for which \( p > 0 \), does \( S_n \) converge? Why?
   (b) Dane tosses a heavily biased coin repeatedly, with independent outcomes. He is
   convinced that if he chooses the probability of heads \( p \) to be small enough (say, \( p \approx 10^{-6} \)), then only finitely-many heads will ever appear. Is Dane right? Justify your answer.
(c) Let \( \{X_n\} \) be an iid sequence of random variables with a nondegenerate distribution (i.e., not concentrated on a single point). Show that

\[
P[\omega : X_n(\omega) \text{ converges}] = 0
\]

(d) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables \( \{X_n\} \), there exists constants \( c_n \to \infty \) such that

\[
P\left( \lim_{n \to \infty} \frac{X_n}{c_n} = 0 \right) = 1.
\]

Give a careful description of how you choose \( c_n \). Find a suitable sequence \( \{c_n\} \)
explicitly for an iid sequence \( \{X_n\} \overset{\text{iid}}{\sim} \text{Ex}(1) \) of unit-rate exponentially-distributed random variables to ensure that \( X_n/c_n \to 0 \) almost surely.


(a) Suppose \( \{A_n, n \in \mathbb{N}\} \) are independent events satisfying \( \mathbb{P}(A_n) < 1, \forall n \in \mathbb{N} \). Show that \( \mathbb{P}(\bigcap_{n=1}^{\infty} A_n) = 1 \) if and only if \( \mathbb{P}(A_n \text{ i.o.}) = 1 \) (“i.o.” means “infinitely often”, so the question concerns \( \limsup A_n \)). Give an example to show that the condition \( \mathbb{P}(A_n) < 1 \) cannot be dropped.

(b) Suppose \( \{A_n\} \) is a sequence of events. If \( \mathbb{P}(A_n) \to 1 \) as \( n \to \infty \), prove that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( \mathbb{P}(\cap_k A_{n_k}) > 0 \).

(c) Let \( A_n \) be a sequence of events. If there exists \( \epsilon > 0 \) such that \( \mathbb{P}(A_n) \geq \epsilon \) for all \( n \in \mathbb{N} \), does it follow that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( \mathbb{P}(\cap_k A_{n_k}) > 0 \)? Why or why not?

(d) Let \( \{X_n\} \) be non-negative iid random variables, with tail \( \sigma \)-field

\[
\mathcal{T} \equiv \bigcap_n \mathcal{F}_n', \quad \mathcal{F}_n' \equiv \sigma\{X_m : m \geq n\}
\]

Is the event

\[
E = \{\text{There exists } \epsilon > 0 \text{ such that } X_n \geq n\epsilon \text{ for infinitely-many } n\}
\]

\[
= \bigcup_{\epsilon > 0} \bigcap_{m \geq 1} \bigcup_{n \geq m} \{\omega : X_n(\omega) \geq n\epsilon\}
\]

in \( \mathcal{T} \)? Prove or disprove it.

Express the probability \( \mathbb{P}[E] \) in terms of the random variables’ common distribution—for example, using their common CDF \( F(x) \equiv \mathbb{P}[X_n \leq x] \) or moments \( \mathbb{E}[X_n^p] \) for some \( p \in \mathbb{R} \).