Sta 711 : Homework 5

1. **Indepedence**.

   (a) Let $\{B_i\}$ be independent events. For $n \in \mathbb{N}$ show that

   $$\mathsf{P}\left(\bigcup_{i=1}^{n} B_i\right) = 1 - \prod_{i=1}^{n}[1 - \mathsf{P}(B_i)] \geq 1 - \exp\left\{-\sum_{i=1}^{n} \mathsf{P}(B_i)\right\}$$

   (b) If $\{A_n, n \in \mathbb{N}\}$ is a sequence of events such that

   $$\mathsf{P}(A_n \cap A_m) = \mathsf{P}(A_n)\mathsf{P}(A_m)\ \forall n, m \in \mathbb{N},\ n \neq m,$$

   does it follow that the events $\{A_n\}$ are independent? Give a proof or counter-example.

   (c) Let Y be a random variable. Show that Y is independent of itself if and only if, for some constant $c \in \mathbb{R}$, $\mathsf{P}[Y = c] = 1$.
   Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, and X any random variable. Can $f(X)$ and X be independent? Explain your answer.

   (d) Give an example to show that an event $A \in \mathcal{F}$ may be independent of each B in some collection $\mathcal{C} \subset \mathcal{F}$ of events, but *not* independent of $\sigma(\mathcal{C})$. Prove this is impossible if $\mathcal{C}$ is a π-system (*i.e.*, in that case A must be independent of $\sigma(\mathcal{C})$).

   (e) Give a simple example to show that two random variables on the same space $(\Omega, \mathcal{F})$ may be independent according to one probability measure P_1 but dependent with respect to another P_2.

2. **Borel Cantelli**.

   (a) Let $\{X_n\}$ be a sequence of Bernoulli random variables with

   $$\mathsf{P}(X_n = 1) = n^{-p} \qquad \mathsf{P}(X_n = 0) = 1 - n^{-p}$$

   for some $p > 0$. For $p = 2$ show that the partial sum

   $$S_n := \sum_{k=1}^{n} X_k$$

   converges almost-surely, whether or not the $\{X_n\}$ are independent. If the $\{X_n\}$ are independent, for which $p > 0$, does S_n converge? Why?

   (b) Dane tosses a heavily biased coin repeatedly, with independent outcomes. He is convinced that if he chooses the probability of heads p to be small enough (say, $p \approx 10^{-6}$), then only finitely-many heads will ever appear. Is Dane right? Justify your answer.

(c) Let $\{X_n\}$ be an iid sequence of random variables with a nondegenerate distribution (*i.e.*, not concentrated on a single point). Show that

$$\mathsf{P}[\omega: \ X_n(\omega) \text{ converges}] = 0$$

(d) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables $\{X_n\}$, there exists constants $c_n \to \infty$ such that

$$\mathsf{P}\left(\lim_{n\to\infty} \frac{X_n}{c_n} = 0\right) = 1.$$

Give a careful description of how you choose c_n. Find a suitable sequence $\{c_n\}$ explicitly for an iid sequence $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ of unit-rate exponentially-distributed random variables to ensure that $X_n/c_n \to 0$ almost surely.

3. **Mixed Bag**.

(a) Suppose $\{A_n, n \in \mathbb{N}\}$ are independent events satisfying $\mathsf{P}(A_n) < 1$, $\forall n \in \mathbb{N}$. Show that $\mathsf{P}(\bigcup_{n=1}^{\infty} A_n) = 1$ if and only if $\mathsf{P}(A_n \text{ i.o.}) = 1$ ("i.o." means "infinitely often", so the question concerns $\limsup A_n$). Give an example to show that the condition $\mathsf{P}(A_n) < 1$ cannot be dropped.

(b) Suppose $\{A_n\}$ is a sequence of events. If $\mathsf{P}(A_n) \to 1$ as $n \to \infty$, prove that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathsf{P}(\cap_k A_{n_k}) > 0$.

(c) Let A_n be a sequence of events. If there exists $\epsilon > 0$ such that $\mathsf{P}(A_n) \geq \epsilon$ for all $n \in \mathbb{N}$, does it follow that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathsf{P}(\cap_k A_{n_k}) > 0$? Why or why not?

(d) Let $\{X_n\}$ be non-negative iid random variables, with tail σ-field

$$\mathcal{T} \equiv \bigcap_n \mathcal{F}'_n, \qquad \mathcal{F}'_n \equiv \sigma\{X_m: \ m \geq n\}$$

Is the event

$$\begin{aligned} E &= \{\text{There exists } \epsilon > 0 \text{ such that } X_n \geq n\epsilon \text{ for infinitely-many } n\} \\ &= \bigcup_{\epsilon>0} \bigcap_{m\geq 1} \bigcup_{n \geq m} \{\omega: \ X_n(\omega) \geq n\,\epsilon\} \end{aligned}$$

in $\mathcal{T}$? Prove or disprove it.

Express the probability $\mathsf{P}[E]$ in terms of the random variables' common distribution— for example, using their common CDF $F(x) \equiv \mathsf{P}[X_n \leq x]$ or moments $\mathsf{E}[{X_n}^p]$ for some $p \in \mathbb{R}$.