Sta 711: Homework 7

Fubini’s Theorem

1. Let \( X \geq 0 \) be a positive random variable and \( \alpha > 0 \). Show that

\[
E(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1} P(X > t) dt.
\]

Note that the distribution \( \mu(dx) \) of \( X \) need not be absolutely continuous. Where did you use Fubini’s theorem?

Uniform Integrability (UI)

2. Let \( \{X_n\} \) be an iid sequence of \( L_1 \) random variables. Set \( S_n = \sum_{i=1}^n X_i \). Show that the sequence of random variables \( \{Y_n\} \) defined by \( Y_n = S_n/n \) is UI.

3. Let \( X_n \sim \text{No}(0, \sigma_n^2) \). Find a simple (easily verifiable) condition on \( \{\sigma_n^2\} \) such that \( \{X_n\} \) is UI.

4. If \( \{X_n\} \) and \( \{Y_n\} \) are UI, show that so is \( \{X_n + Y_n\} \).

5. Let \( \phi(x) \) be a function which grows faster than \( x \) at infinity, i.e., \( \phi(x)/x \rightarrow \infty \) as \( x \rightarrow \infty \). Let \( \mathcal{C} \) be a collection of random variables such that, for some fixed \( B < \infty \) and all \( Z \in \mathcal{C} \),

\[
E\left(\phi(|Z|)\right) \leq B.
\]

Show that \( \mathcal{C} \) is UI. Note: This implies any collection of random variables that is bounded in \( L_p \) for some \( p > 1 \) is UI.

Convergence Theorems Revisited

6. Let \( X \) be a non-negative real valued random variable. Show that:

   (a) \( \lim_{n \to \infty} nE(\frac{1}{X} 1_{[X > n]} ) = 0. \)

   (b) \( \lim_{n \to \infty} n^{-1}E(\frac{1}{X} 1_{[X > n-1]} ) = 0. \)

7. Let \( \{p_k\} \) be a probability mass function on \( \mathbb{N}_0 = \{0,1,...\} \) and define the generating function

\[
G(z) \equiv \sum_{k=0}^\infty p_k z^k \quad 0 \leq z \leq 1
\]
Use the Dominated Convergence Theorem to prove that

\[ \frac{d}{dz} G(z) = \sum_{k=0}^{\infty} p_k k z^{k-1} \quad 0 \leq z < 1. \tag{1} \]

What is \( G(0) \)? \( G'(0) \)? \( G'(1) \)? How can you find each \( p_k \) explicitly from \( G(z) \)?

Extra Credit (+1pt):
Show that (1) also holds at \( z = 1 \), if we interpret \( \frac{d}{dz} G(z) \) as the left derivative \( G'(1-) \equiv \lim_{\epsilon \downarrow 0} [G(1) - G(1 - \epsilon)]/\epsilon \).