Sta 711: Homework 8

Uniform Integrability

1. **True or false?** Answer whether each of the following statements is true or false. If true, answer why; if false, give a simple counter example.

   (a) If $\{X_n,\ n \in \mathbb{N}\}$ is a uniformly integrable (UI) collection of random variables, then $X_n \in L_1$ for each n.

   (b) Define a sequence $\{X_n\}$ of random variables on the unit interval with Lebesgue measure, $(\Omega, \mathcal{F}, P)$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, and $\mathsf{P} = \lambda$, by $X_n \equiv \sqrt{n}\mathbf{1}_{(0,\frac{1}{n}]}$. Then $\{X_n\}$ is UI.

   (c) Let $\{X_n\}$ be a sequence of random variables for which $e^{|X_n|}$ is uniformly bounded in L_1, *i.e.*, satisfies $\mathsf{E}e^{|X_n|} \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

   (d) Let $\{X_n\}$ be a sequence of random variables that is uniformly bounded in L_1, *i.e.*, satisfies $\mathsf{E}|X_n| \le B$ for some $B < \infty$ and all n. Then $\{X_n\}$ is UI.

Characteristic Functions

2. Let X be a random variable, and define

$$\phi_X(\omega) \equiv \mathsf{E}(e^{i\omega X}), \qquad \omega \in \mathbb{R}$$

   Show that $\phi_X(\omega)$ is uniformly continuous in $\mathbb{R}$.

3. Find the characteristic functions of the following random variables:

   (a) $W \equiv c$[1] (The superscripts in (a)–(c) are footnote indicators, not exponents)

   (b) $X \sim \mathsf{Un}(a, b)$[2]

   (c) $Y \sim \mathsf{Ga}(\alpha, \lambda)$[3]

   (d) $Z = (Y_1 + Y_2 + \cdots + Y_n)/n, \quad Y_j \overset{\text{iid}}{\sim} \mathsf{Ga}(\alpha, \lambda)$

   What is the distribution of Z? What happens as $n \to \infty$?

4. The distribution of a random variable X is called *infinitely divisible* if, for every $n \in \mathbb{N}$, there exist n iid random variables $\{Y_i\}$ such that X has the same distribution as $\sum_{i=1}^n Y_i$. Use characteristic functions to show that if $X \sim \mathsf{Po}(\lambda)$, then X is infinitely divisible.[4]

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[1]A constant random variable with value $c \in \mathbb{R}$

[2]Uniform, on the interval $(a, b) \subset \mathbb{R}$

[3]Gamma, with rate parameterization— with pdf $f(y \mid \lambda) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y}/\Gamma(\alpha)$, $y > 0$.

[4]Hint: If $\{Y_i\}$ are independent with sum $Y_+ := \sum Y_i$, then $\phi_{Y_+}(\omega) = \prod \phi_{Y_i}(\omega)$ for all $\omega \in \mathbb{R}$.