Sta 711: Homework 8

Uniform Integrability

1. True or false? Answer whether each of the following statements is true or false. If true, answer why; if false, give a simple counter example.

   (a) If \( \{X_n, n \in \mathbb{N}\} \) is a uniformly integrable (UI) collection of random variables, then \( X_n \in L_1 \) for each \( n \).

   (b) Define a sequence \( \{X_n\} \) of random variables on the unit interval with Lebesgue measure, \( (\Omega, \mathcal{F}, P) \) with \( \Omega = (0,1] \), \( \mathcal{F} = \mathcal{B} \), and \( P = \lambda \), by \( X_n \equiv \sqrt{n} I_{(0, \frac{1}{n})} \). Then \( \{X_n\} \) is UI.

   (c) Let \( \{X_n\} \) be a sequence of random variables for which \( e^{X_n} \) is uniformly bounded in \( L_1 \), i.e., satisfies \( E(e^{X_n}) \leq B \) for some \( B < \infty \) and all \( n \). Then \( \{X_n\} \) is UI.

   (d) Let \( \{X_n\} \) be a sequence of random variables that is uniformly bounded in \( L_1 \), i.e., satisfies \( E|X_n| \leq B \) for some \( B < \infty \) and all \( n \). Then \( \{X_n\} \) is UI.

Characteristic Functions

2. Let \( X \) be a random variable, and define

\[
\phi_X(\omega) \equiv E(e^{i\omega X}), \quad \omega \in \mathbb{R}
\]

Show that \( \phi_X(\omega) \) is uniformly continuous in \( \mathbb{R} \).

3. Find the characteristic functions of the following random variables:

   (a) \( W \equiv c^1 \) (The superscripts in (a)-(c) are footnote indicators, not exponents)

   (b) \( X \sim \text{Un}(a, b)^2 \)

   (c) \( Y \sim \text{Ga}(\alpha, \lambda)^3 \)

   (d) \( Z = (Y_1 + Y_2 + \cdots + Y_n)/n, \quad Y_j \overset{iid}{\sim} \text{Ga}(\alpha, \lambda) \)

What is the distribution of \( Z \)? What happens as \( n \to \infty \)?

4. The distribution of a random variable \( X \) is called infinitely divisible if, for every \( n \in \mathbb{N} \), there exist \( n \) iid random variables \( \{Y_i\} \) such that \( X \) has the same distribution as \( \sum_{i=1}^n Y_i \). Use characteristic functions to show that if \( X \sim \text{Po}(\lambda) \), then \( X \) is infinitely divisible.\(^4\)

\(^1\) A constant random variable with value \( c \in \mathbb{R} \)

\(^2\) Uniform, on the interval \( (a, b) \subset \mathbb{R} \)

\(^3\) Gamma, with rate parameterization—with pdf \( f(y \mid \lambda) = \lambda^y y^{\alpha-1} e^{-\lambda y} / \Gamma(\alpha), y > 0 \).

\(^4\) Hint: If \( \{Y_i\} \) are independent with sum \( Y_n := \sum Y_i \), then \( \phi_{Y_n}(\omega) = \prod \phi_{Y_i}(\omega) \) for all \( \omega \in \mathbb{R} \).