Midterm Examination I

STA 711: Probability & Measure Theory

Thursday, 2012 Oct 11, 11:45 am – 1:00pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss questions with others.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1. Let $\Omega = \{a, b, c, d\}$ have just four points, with σ-algebra $\mathcal{F} = 2^\Omega$ and probability assignment $\mathsf{P}[A] = \sum_{i=1}^4 \frac{i}{10}\mathbf{1}_A(\omega_i)$ to events $A \in \mathcal{F}$, where $\omega_1 = a$, $\omega_2 = b$, $\omega_3 = c$ and $\omega_4 = d$. Define a collection of sets by

$$\mathcal{G} = \big\{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{a, d\}, \{b, c, d\}, \Omega\big\}$$

a) (8) Give explicitly a real random variable X that generates $\mathcal{G} = \sigma(X)$:

$X(a) =$ ____ $X(b) =$ ____ $X(c) =$ ____ $X(d) =$ ____

b) (6) Give explicitly a real random variable Y that takes only two distinct values, for which $\mathcal{F} = \sigma(X, Y)$:

$Y(a) =$ ____ $Y(b) =$ ____ $Y(c) =$ ____ $Y(d) =$ ____

c) (6) Find the expectation of your random variables X and Y above:

$\mathsf{E}[X] =$ ______________ $\mathsf{E}[Y] =$ ______________

Problem 2. Again let $\Omega = \{a, b, c, d\}$ with $\mathcal{F} = 2^\Omega$ and P that assigns probabilities 1/10, 2/10, and 3/10 respectively to the singleton sets $\{a\}$, $\{b\}$ and $\{c\}$. Consider the two fields

$$\mathcal{C}_1 = \{\emptyset,\ \{a, b\},\ \{c, d\},\ \Omega\}$$
$$\mathcal{C}_2 = \{\emptyset,\ \{a, c\},\ \{b, d\},\ \Omega\}$$

a) (8) Are $\mathcal{C}_1$ and $\mathcal{C}_2$ independent? Give a proof or counterexample.
Y N Why?

b) (6) Find a real random variable X that is $\mathcal{C}_2\backslash\mathcal{B}$-measurable but *not* $\mathcal{C}_1\backslash\mathcal{B}$-measurable (be careful not to mix up 1 and 2).

$X(a) =$ ____ $X(b) =$ ____ $X(c) =$ ____ $X(d) =$ ____

c) (6) Find all random variables that are *both* $\mathcal{C}_2\backslash\mathcal{B}$ *and* $\mathcal{C}_1\backslash\mathcal{B}$-measurable. Justify your answer.

Problem 3. Let $\{U_n\}$ be independent random variables with uniform distributions on $(0,1]$ and let $\{p_n\}$ be (non-random) numbers in $(0,1)$. Set:

$$X_n = \mathbf{1}_{\{U_n \le p_n\}} \qquad Y_N = \prod_{1\le n\le N} X_n,$$

each taking the values 0 or 1.

a) (8) What is the *probability distribution* of X_n?
$\mu_{X_n}(B) =$

b) (6) If possible, find and box a sequence $\{p_n\}$ for which you can show the event

$$\left[\lim_{N\to\infty} Y_N > 0\right]$$

has positive probability; if this is not possible, explain why.

c) (6) If possible, find and box a sequence $\{p_n\}$ for which you can show the event

$$\left[\sum_{N=1}^{\infty} X_N < \infty\right]$$

has positive probability; if this is not possible, explain why.

Problem 4. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the natural numbers $\Omega = \mathbb{N} = \{1, 2, 3, \dots\}$ with $\mathcal{F} = 2^\Omega$ and $\mathsf{P}[A] = \sum\{2^{-\omega} : \ \omega \in A \cap \mathbb{N}\}$.

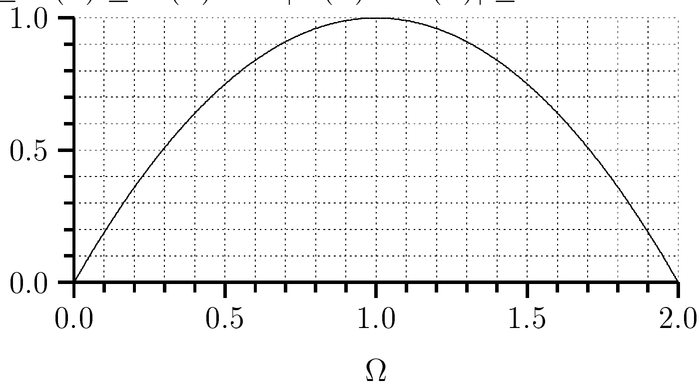
a) (7) Fix $\lambda \in \mathbb{R}$. Is the random variable $X(\omega) = e^{\lambda\omega}$ in $L_1(\Omega, \mathcal{F}, \mathsf{P})$? If so, find $\mathsf{E}[X]$ in closed form; if not, tell why; if this depends on λ, explain.
◯ Yes ◯ No ◯ It Depends Reasoning?

b) (7) For $n \in \mathbb{N}$ define a random variable Y_n by $Y_n(\omega) = n$ if $\omega \ge n$, $Y_n(\omega) = 0$ if $\omega < n$. Does the Dominated Convergence Theorem apply to $\{Y_n\}$? If so, tell what DCT says and show why it applies; if not, explain why.
◯ Yes ◯ No Reasoning:

c) (6) Define Y_n as above. Does Fatou's Lemma apply? If so, verify Fatou's conclusion **by calculation**; if not, why? ◯ Yes ◯ No Reasoning:

Problem 5. Let $X = \omega(2-\omega)$ be a random variable on the space $\Omega = (0, 2]$ with $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel sets (it's plotted below).

a) (7) Find *and plot* a non-negative *simple* random variable $Y \in \mathcal{E}_+$ satisfying $0 \le Y(\omega) \le X(\omega)$ and $|X(\omega) - Y(\omega)| \le 0.4$ for all $\omega \in \Omega$.



$Y(\omega) =$

b) (7) Find $\mathsf{E}X$ and $\mathsf{E}Y$ for the probability measure $\mathsf{P}(d\omega) = d\omega/2$ (*i.e.*, $\mathsf{P}\{(a, b]\} = (b - a)/2$ for all $0 \le a \le b \le 2$):

$\mathsf{E}X =$ ____________ $\mathsf{E}Y =$ ____________

c) (6) Let $Z = 1_{(0,1]}(\omega)$. Are X and Z independent on $(\Omega, \mathcal{F}, \mathsf{P})$?
◯ Yes ◯ No Why?

Blank Worksheet

Another Blank Worksheet