

Sta 711: Homework 8

Uniform Integrability

1. **True or false?** Answer whether each of the following statements is true or false. If true, answer why; if false, give a simple counter example.
 - (a) If $\{X_n, n \in \mathbb{N}\}$ is a uniformly integrable (UI) collection of random variables, then $X_n \in L_1$ for each n .
 - (b) Define a sequence $\{X_n\}$ of random variables on the unit interval with Lebesgue measure, (Ω, \mathcal{F}, P) with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, and $P = \lambda$, by $X_n \equiv \sqrt{n} \mathbf{1}_{(0, \frac{1}{n}]}$. Then $\{X_n\}$ is UI.
 - (c) Let $\{X_n\}$ be a sequence of random variables for which $e^{|X_n|}$ is uniformly bounded in L_1 , *i.e.*, satisfies $\mathbb{E}e^{|X_n|} \leq B$ for some $B < \infty$ and all n . Then $\{X_n\}$ is UI.
 - (d) Let $\{X_n\}$ be a sequence of random variables that is uniformly bounded in L_1 , *i.e.*, satisfies $\mathbb{E}|X_n| \leq B$ for some $B < \infty$ and all n . Then $\{X_n\}$ is UI.

Characteristic Functions

2. Let X be a random variable, and define

$$\phi_X(\omega) \equiv \mathbb{E}(e^{i\omega X}), \quad \omega \in \mathbb{R}$$

Show that $\phi_X(\omega)$ is uniformly continuous in \mathbb{R} .

3. Find the characteristic functions of the following random variables:

- (a) $W \equiv c^1$ (The superscripts in (a)–(c) are footnote indicators, not exponents)
- (b) $X \sim \text{Un}(a, b)^2$
- (c) $Y \sim \text{Ga}(\alpha, \lambda)^3$
- (d) $Z = (Y_1 + Y_2 + \cdots + Y_n)/n, \quad Y_j \stackrel{\text{iid}}{\sim} \text{Ga}(\alpha, \lambda)$

What is the distribution of Z ? What happens as $n \rightarrow \infty$?

4. The distribution of a random variable X is called *infinitely divisible* if, for every $n \in \mathbb{N}$, there exist n iid random variables $\{Y_i\}$ such that X has the same distribution as $\sum_{i=1}^n Y_i$. Use characteristic functions to show that if $X \sim \text{Po}(\lambda)$, then X is infinitely divisible.⁴

¹A constant random variable with value $c \in \mathbb{R}$

²Uniform, on the interval $(a, b) \subset \mathbb{R}$

³Gamma, with rate parameterization— with pdf $f(y | \lambda) = \lambda^\alpha y^{\alpha-1} e^{-\lambda y} / \Gamma(\alpha)$, $y > 0$.

⁴Hint: If $\{Y_i\}$ are independent with sum $Y_+ := \sum Y_i$, then $\phi_{Y_+}(\omega) = \prod \phi_{Y_i}(\omega)$ for all $\omega \in \mathbb{R}$.