Midterm Examination I

STA 711: Probability & Measure Theory

Thursday, 2013 Oct 3, 11:45 am – 1:00pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss questions with others.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. For full credit, give answers in **closed form** (no unevaluated sums, integrals, maxima, *etc.*) where possible and **simplify**. Good luck!

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Print Name: ____________________

Problem 1: Let $\Omega = \{a, b, c, d\}$ have just four points, with σ-algebra $\mathcal{F} = 2^\Omega$ and probability assignment $\mathsf{P}[A] = \sum_{i=1}^{4} \frac{i}{10} \mathbf{1}_A(\omega_i)$ to events $A \in \mathcal{F}$, where $\omega_1 = a$, $\omega_2 = b$, $\omega_3 = c$ and $\omega_4 = d$. Define a collection of sets by

$$\mathcal{G} = \big\{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \Omega\big\}$$

a) (8) Is $\mathcal{G}$ a λ-system? If *yes*, state and illustrate what conditions need to be verified (you don't have to verify these conditions for every possible combination of sets); if *no*, explain why not.

Choose one: ◯ Yes ◯ No Reasoning:

b) (4) Find the σ-algebra generated by $\mathcal{G}$:

$$\sigma(\mathcal{G}) = \left\{ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \right\}$$

c) (4) Give a real RV Y on $(\Omega, \mathcal{F}, \mathsf{P})$ that is *not* $\sigma(\mathcal{G})$-measurable, if possible, and explain why; if this isn't possible, explain why not.

$Y(a) =$ ___ $Y(b) =$ ___ $Y(c) =$ ___ $Y(d) =$ ___

d) (4) Find the expectation of your random variable Y:

$\mathsf{E}[Y] =$ ___________

Problem 2: Again let $\Omega = \{a, b, c, d\}$ with $\mathcal{F} = 2^{\Omega}$ and P that assigns probabilities 1/10, 2/10, 3/10 and 4/10 respectively to the singleton sets $\{a\}$, $\{b\}$, $\{c\}$ and $\{d\}$. Consider the random variables

$$\begin{array}{llll} X(a)=1 & X(b)=2 & X(c)=1 & X(d)=0 \\ Y(a)=0 & Y(b)=1 & Y(c)=0 & Y(d)=1 \\ Z(a)=0 & Z(b)=0 & Z(c)=1 & Z(d)=1 \end{array}$$

a) (9) Find the σ-algebras generated by each:

$\sigma(X) = \left\{ \qquad \right\}$

$\sigma(Y) = \left\{ \qquad \right\}$

$\sigma(Z) = \left\{ \qquad \right\}$

b) (6) Which (if any) of the eight possible collections $\mathcal{C} \subset \{X, Y, Z\}$ generate $\mathcal{F} = \sigma(\mathcal{C})$? Enumerate them.
$\mathcal{C} =$

c) (5) Are $\{Y, Z\}$ independent? ◯ Yes ◯ No Reasoning:

Problem 3: Let $\Omega = (0,1]$ with the Borel sets $\mathcal{F} = \mathcal{B}(\Omega)$ and Lebesgue measure $\mathsf{P} = \lambda$. Consider the random variables:

$$X_n(\omega) := \sqrt{n}\mathbf{1}_{\{\omega<1/n\}} \qquad\qquad Y_n(\omega) := \frac{1}{2\sqrt{n\omega}}$$

a) (4) Find the indicated expectations (simplify!):

$\mathsf{E}[X_n] =$ ________________ $\qquad$ $\mathsf{E}[Y_n] =$ ________________

b) (10) Prove that for each ω, $X_n \to 0$ and $Y_n \to 0$, as follows. For each $0 < \epsilon < 1$, find the smallest $N_\epsilon(\omega)$ such that:

$n \geq N_\epsilon \Rightarrow |X_n(\omega)| \leq \epsilon :\quad N_\epsilon(\omega) =$ ________________

$n \geq N_\epsilon \Rightarrow |Y_n(\omega)| \leq \epsilon :\quad N_\epsilon(\omega) =$ ________________

c) (6) For each $n \in \mathbb{N}$, find the indicated probabilities:

$\mathsf{P}[X_n \geq 10] =$ ________________

$\mathsf{P}[Y_n \geq 10] =$ ________________

$\mathsf{P}[Y_n \geq X_n] =$ ________________

Problem 4: Let $\{U_n\}$ be independent random variables with uniform distributions on $(0,1]$ and set:

$$X_n := \mathbf{1}_{\{U_n \le 1/2\}} \qquad Y_n := \min_{1 \le m \le n} X_m,$$

each taking only the values 0 and 1.

a) (4) What is the *probability distribution* of Y_n? For $B \in \mathcal{B}(\mathbb{R})$,

$$\mu_{Y_n}(B) = \begin{cases} \\ \\ \end{cases}$$

b) (8) Find the almost-sure quanities:

$$\liminf_{n\to\infty} X_n = \underline{\qquad\qquad} \qquad \liminf_{n\to\infty} Y_n = \underline{\qquad\qquad}$$

$$\limsup_{n\to\infty} X_n = \underline{\qquad\qquad} \qquad \limsup_{n\to\infty} Y_n = \underline{\qquad\qquad}$$

c) (8) Fix $\lambda > 1$. For which (if any) $p > 0$ does

$$Z_n := \lambda^n Y_n$$

converge to zero in L_p? Why?

Problem 5: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the nonnegative numbers $\Omega = \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ with $\mathcal{F} = 2^\Omega$ and $\mathsf{P}[A] := e^{-1} \sum_{\omega \in A} \frac{1}{\omega!}$ for $A \in \mathcal{F}$.

a) (7) Fix $p > 0$. Is the random variable $X(\omega) = 2^\omega$ in $L_p(\Omega, \mathcal{F}, \mathsf{P})$? If so, find $\|X\|_p$ in closed form. If not, tell why. If this depends on p, explain.

◯ Yes ◯ No ◯ It Depends Reasoning?

$\|X\|_p =$ ________________

b) (6) Is $Z(\omega) := \omega$ in $L_1(\Omega, \mathcal{F}, \mathsf{P})$? If so, find $\mathsf{E}Z$ (a numerical answer). If not, explain. ◯ Yes ◯ No Reasoning:

$\mathsf{E}Z =$ ________________

c) (7) For $n \in \mathbb{N}$ define a random variable Y_n by $Y_n(\omega) = n$ if $\omega > n$, $Y_n(\omega) = 0$ if $\omega \le n$. Does the Dominated Convergence Theorem apply to $\{Y_n\}$? If so, tell what DCT says and show why it applies; if not, explain why.

◯ Yes ◯ No Reasoning:

d) (XC) Find $S := \sum Y_n$ and $\mathsf{E}[S]$.

Blank Worksheet

Another Blank Worksheet