Midterm Examination I

STA 711: Probability & Measure Theory

Wednesday, 2014 Oct 1, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this.

It is to your advantage to write your solutions as clearly as possible, and to box answers I might not find.

For full credit, give answers in **closed form** (no unevaluated sums, integrals, maxima, unreduced fractions, *etc.*) where possible and **simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $\{A_i : i \in \mathbb{N}\} \subset \mathcal{F}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

a) (5) Call $\{A_i\}$ "almost disjoint" if $(i \neq j) \Rightarrow \mathsf{P}[A_i \cap A_j] = 0$. If $\{A_i\}$ are almost disjoint, prove that $\mathsf{P}[\cup_{i\in\mathbb{N}} A_i] = \sum_{i\in\mathbb{N}} \mathsf{P}[A_i]$.

b) (5) If $\mathsf{P}[\cup_{i\in\mathbb{N}} A_i] = \sum_{i\in\mathbb{N}} \mathsf{P}[A_i]$, prove that $\{A_i\}$ are almost disjoint.

Problem 1 (cont'd): As before, $\{A_i : i \in \mathbb{N}\} \subset \mathcal{F}$ on $(\Omega, \mathcal{F}, \mathsf{P})$.

c) (5) If $A_i \supset A_{i+1}$ for each $i \in \mathbb{N}$, does it follow that $\mathsf{P}[\cap_{i\in\mathbb{N}} A_i] = \lim_{i\to\infty} \mathsf{P}[A_i]$? Give a proof or a counter-example.

d) (5) If $\mathsf{P}[\cap_{i\in\mathbb{N}} A_i] = \lim_{i\to\infty} \mathsf{P}[A_i]$, does it follow that $A_i \supset A_{i+1}$ for each $i \in \mathbb{N}$? Give a proof or a counter-example.

Problem 2: Let $\Omega := \{1,2,3,4\}$, $\mathcal{F} := 2^\Omega$, and $\mathsf{P}(A) := \sum_{\omega\in A} \frac{12}{25\omega}$ for $A \in \mathcal{F}$. Set $X(\omega) := \omega$ for $\omega \in \Omega$.

a) (4) Verify that $\mathsf{P}(\Omega) = 1$.

b) (8) Find: $\mathsf{E}X =$ $\mathsf{E}X^2 =$

c) (4) Find: $\mathsf{P}[X > 2.5] =$

d) (4) Find: $\mathsf{E}[X!] =$

Problem 3: Let $\Omega = \mathbb{R}$ be the real line, and consider the class of subsets

$$\mathcal{G} = \{(a, b) : \ -\infty < a \le b < \infty\}$$

and let $\mathcal{F} = \mathcal{F}(\mathcal{G})$ be the field generated by $\mathcal{G}$.

a) (5) Is $\mathcal{G}$ a π-system? Prove your answer.

b) (5) Is the one-point set $\{0\}$ in the field $\mathcal{F}$? Prove your answer.

c) (5) Find a set $A \in \sigma(\mathcal{G})$ that is *not* in $\mathcal{F}$. Briefly, why isn't it in $\mathcal{F}$?

d) (5) Find a random variable X taking at least three distinct values that is $\mathcal{F}$-measurable, *i.e.*, satisfies $\mathcal{F}_X := X^{-1}(\mathcal{B}) \subset \mathcal{F}$.

Problem 4: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space with $\Omega = \mathbb{R}$ the real line, $\mathcal{F}$ the Borel sets, and probability measure P given on the π-system $\mathcal{P} = \{(a, \infty): \ a > 0\}$ of open semi-infinite positive intervals by

$$\mathsf{P}[(a, \infty)] = e^{-a^2}.$$

a) (5) Find a function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\mathsf{P}[(a, \infty)] = \int_a^\infty \phi(x)\, dx$ for every $a \in \mathbb{R}$. Give $\phi(x)$ correctly for every $x \in (-\infty, \infty)$.

b) (5) Find the probablity distribution[1] $\mu_Y(dy)$ for $Y(\omega) := \omega^2$.

[1] You must tell how to compute $\mu_Y(B)$ for every Borel $B \subset \mathbb{R}$. Hint: What is the event $\{\omega : \ Y(\omega) > y\}$ for $y > 0$?

Problem 4 (cont'd): Still $\mathsf{P}[(a,\infty)] = e^{-a^2}$ for $a > 0$, on $\Omega = \mathbb{R}$.

c) (4) Find $\mathsf{P}[(-1,1)]$.

d) (6) For all $p > 0$, evaluate[2] $\|X\|_p$ for the random variable $X(\omega) := \omega$.

[2]Recall $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx$ for $z > 0$.

Problem 5: True or false? Circle one; each answer is 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky. All random variables are real on the same $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F Always $\|X\|_2 \le \|X\|_5$.

b) T F For any RV Y and any $a > 0$, $\mathsf{P}[Y \ge a] \le \mathsf{E}[Y^4]/a^4$.

c) T F For any RVs $\{X_n\}$ with $|X_n| \le Y$ for some $Y \in L_1(\Omega, \mathcal{F}, \mathsf{P})$,

$$\mathsf{E}\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} \big[\mathsf{E} X_n\big].$$

d) T F If $\{X_n\}$ are independent, then also $\{X_n^2\}$ are independent.

e) T F If $X_n(\omega) \to 0$ as $n \to \infty$ for every $\omega \in \Omega$, then $\mathsf{E}[X_n] \to 0$.

f) T F Let $X > 0$ for every ω and set $Y := 1/X$. Then $\mathsf{E}[X] \le 1/\mathsf{E}[Y]$.

g) T F If X, Y, and Z are all in $L_3(\Omega, \mathcal{F}, \mathsf{P})$ then $XYZ \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ with

$$\|XYZ\|_1 \le \|X\|_3 \, \|Y\|_3 \, \|Z\|_3.$$

h) T F $X \in L_4(\Omega, \mathcal{F}, \mathsf{P})$ if and only if $X^2 \in L_2(\Omega, \mathcal{F}, \mathsf{P})$.

i) T F If $X = f(Z)$ and $Y = g(Z)$ for Borel functions f, g, then X and Y cannot be independent with non-trivial distributions, since they're both functions of the same random variable Z.

j) T F $\mathcal{T} := \{A \in \mathcal{F} : \ \mathsf{P}[A] = 0 \text{ or } \mathsf{P}[A] = 1\}$ is a σ-algebra.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha, \beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$x \in (0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n, p)$	$f(x) = \binom{n}{x} p^x q^{(n-x)}$	$x \in 0, \cdots, n$	$n\,p$	$n\,p\,q$	$(q = 1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x) = \lambda\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha, \lambda)$	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\lambda x}$	$x \in \mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x) = p\,q^x$	$x \in \mathbb{Z}_+$	q/p	q/p^2	$(q = 1-p)$
		$f(y) = p\,q^{y-1}$	$y \in \{1, ...\}$	$1/p$	q/p^2	$(y = x+1)$
HyperGeo.	$\mathsf{HG}(n, A, B)$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x \in 0, \cdots, n$	$n\,P$	$n\,P\,(1-P)\,\frac{N-n}{N-1}$	$(P = \frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu, \beta)$	$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x \in \mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu, \sigma^2)$	$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\log x - \mu)^2/2\sigma^2}$	$x \in \mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha, p)$	$f(x) = \binom{x+\alpha-1}{x} p^\alpha\, q^x$	$x \in \mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q = 1-p)$
		$f(y) = \binom{y-1}{y-\alpha} p^\alpha\, q^{y-\alpha}$	$y \in \{\alpha, ...\}$	α/p	$\alpha q/p^2$	$(y = x+\alpha)$
Normal	$\mathsf{No}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$	$x \in \mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha, \epsilon)$	$f(x) = \alpha\,\epsilon^\alpha / x^{\alpha+1}$	$x \in (\epsilon, \infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$	$x \in \mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1, \nu_2)$	$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1-2}{2}} \left[1 + \frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x \in \mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1 + x^2/\nu]^{-(\nu+1)/2}$	$x \in \mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a, b)$	$f(x) = \frac{1}{b-a}$	$x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha, \beta)$	$f(x) = \alpha\beta\, x^{\alpha-1}\, e^{-\beta\, x^\alpha}$	$x \in \mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha) - \Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	