Midterm Examination II

STA 711: Probability & Measure Theory

Wednesday, 2014 Nov 12, 1:25 – 2:40pm

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this.

It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find.

For full credit, give answers in **closed form** (no unevaluated sums, integrals, maxima, unreduced fractions, *etc.*) where possible and **simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Each integer $n \in \mathbb{N}$ has a unique representation[1] $n = i + 2^j$ for integers $j \ge 0$ and $0 \le i < 2^j$. Set $\Omega := (0, 1]$ and let P be Lebesgue measure on $\mathcal{F} = \mathcal{B}(\Omega)$. Define random variables

$$X_n(\omega) := \mathbf{1}_{\{\frac{i}{2^j} < \omega \le \frac{i+1}{2^j}\}}$$

a) (5) For which real numbers $\alpha \in \mathbb{R}$ do the random variables $Y_n := n^\alpha X_n$ converge to zero in probability?

b) (5) For which real numbers $\alpha \in \mathbb{R}$ do the random variables $Y_n := n^\alpha X_n$ converge to zero almost surely?

[1] Namely, $j = \lfloor \log_2 n \rfloor$ and $i = n - 2^j$. Sometimes it's helpful to note that $2^j \le n < 2^{j+1}$.

Problem 1 (cont'd): As before, $X_n(\omega) := \mathbf{1}_{\{i/2^j < \omega \le (i+1)/2^j\}}$ on $\Omega := (0.1]$ with Lebesgue measure $\mathsf{P}(d\omega)$, for $n = i + 2^j$.

c) (5) For which real numbers $\alpha \in \mathbb{R}$ do the random variables $Y_n := n^\alpha X_n$ converge to zero in L_p, for fixed $1 \le p < \infty$?

d) (5) For any integer $j \in \mathbb{N}$, give the best upper and lower bounds you can for

$$\underline{\qquad\qquad} \le \qquad Y(\omega) := \sum_{n=2^j}^{2^{j+1}-1} X_n(\omega) \qquad \le \underline{\qquad\qquad}$$

Problem 2: Let $\{U_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ be iid standard uniform random variables and let $A_n := \{\omega : \ 0 < U_n \le 1/n\}$ for each $n \in \mathbb{N}$.

a) (5) Prove that $X := \sum_{n=1}^{\infty} n \mathbf{1}_{\{A_{n^2}\}}$ is finite almost-surely.

b) (5) Find $\mathsf{E}[X]$.

c) (5) Prove that $Y := \sum_{n=1}^{\infty} n U_n \mathbf{1}_{\{A_n\}}$ is infinite almost-surely.

d) (5) Is $Z := \sum_{n=1}^{\infty} \frac{1}{n} \mathbf{1}_{\{A_n\}}$ finite $(a.s)$ or infinite $(a.s)$? Why?

Problem 3: Let $X \sim \mathsf{Un}\{1,2,3,4,5,6\}$ be the number shown on a fair six-sided die and let $Y_3 := X \pmod 3$ and $Y_4 := X \pmod 4$ be X, modulo[2] three and four (taking values 0–2 and 0–3), respectively. For four points each, find the indicated conditional expectations. No explanations necessary.

a) $\mathsf{E}[X \mid Y_3] = \Bigg\{$

b) $\mathsf{E}[Y_3 \mid X] = \Bigg\{$

c) $\mathsf{E}[X \mid Y_4] = \Bigg\{$

d) $\mathsf{E}[Y_4 \mid X] = \Bigg\{$

e) $\mathsf{E}[Y_3 \mid Y_4] = \Bigg\{$

[2]Reminder: For real numbers $x \in \mathbb{R}$ and $k > 0$, "x modulo k" or "$x \pmod k$" is that real number $y = x - (k \times \lfloor x/k \rfloor) \in [0, k)$ such that $(x - y)$ is evenly divisible by k. For example, $8 \pmod 3 = 2$, $6 \pmod 2 = 0$, $42 \pmod{17} = 8$, $-4 \pmod 3 = 2$, and $\pi \pmod 2$ is about 1.14159 or so.

Problem 4: The random variables $\{X_n\}$ all have the standard $\mathsf{Ex}(1)$ distribution[3], but they are not independent: $\mathsf{E}[X_i X_j] = 1 + (1/2)^{|i-j|}$ for $i, j \in \mathbb{N}$.

a) (6) Find the mean, variance, and covariance indicated[4]

$\mathsf{E}[X_7] =$ ________ $\mathsf{V}[X_7] =$ ________ $\mathsf{Cov}[X_3, X_7] =$ ____________

b) (5) Find the mean and variance of $S_n := \sum_{i=1}^{n} X_i$.

$\mathsf{E}[S_n] =$ ________________ $\mathsf{V}[S_n] =$ ________________

[3]A sheet of info about common distributions is at the back of this exam, on *p.* 11.

[4]Hint: no integration or infinite summation is needed

Problem 4 (cont'd): Still $\{X_n\} \sim \mathsf{Ex}(1)$ with $\mathsf{E}[X_i X_j] = 1 + (1/2)^{|i-j|}$.

c) (5) Prove that $S_n/n \to 1$ in L_2 as $n \to \infty$.

d) (2) Let $\{\xi_i\} \overset{\text{iid}}{\sim} \mathsf{Ga}(\alpha_i, 1)$. Find $\{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{R}$ such that $X_1, X_2 \sim \mathsf{Ex}(1)$ have the required marginal distributions and covariance, if[5]

$$X_1 = \xi_1 + \xi_2 \qquad\qquad X_2 = \xi_2 + \xi_3$$

$\alpha_1 =$ ____________ $\alpha_2 =$ ____________ $\alpha_3 =$ ____________

[5]Reminder: If $Z_j \overset{\text{ind}}{\sim} \mathsf{Ga}(\alpha_j, \beta)$, what is the distribution of $Z_1 + Z_2$? Also, for what $\alpha > 0$ is $\mathsf{Ex}(\beta) = \mathsf{Ga}(\alpha, \beta)$?

Problem 4 (cont'd): Still $\{X_n\} \sim \mathsf{Ex}(1)$ with $\mathsf{E}[X_i X_j] = 1 + (1/2)^{|i-j|}$.

e) (2) Let $\eta_j \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ and $U \sim \mathsf{Un}(0,1)$ all be independent. Find $p \in (0,1)$ such that $X_1, X_2 \sim \mathsf{Ex}(1)$ have the required marginal distributions and covariance, if

$$X_1 = \eta_1 \qquad\qquad X_2 = \eta_1 \mathbf{1}_{\{U \le p\}} + \eta_2 \mathbf{1}_{\{U > p\}}$$

$$p = \underline{\qquad\qquad}$$

In particular, note that the joint distribution of $\{X_n\}$ is not determined by the information given in the Problem 4 statement on *p.* 5, since (for example) $\mathsf{P}[X_1 = X_2]$ is zero in part d) and $p > 0$ in part e).

Problem 5: True or false? Circle one; each answer is 2 points. No explanations are needed, but you can give one if you think the question is ambiguous or tricky. All random variables are real on the same $(\Omega, \mathcal{F}, \mathsf{P})$.

a) T F If $X_n \Rightarrow X$ then $X_{n_k} \to X$ $a.s$ along a subsequence $\{n_k\}$.

b) T F If $X_n \to X$ $(pr.)$ then $\cos(X_n) \to \cos(X)$ in L_2.

c) T F If $\mathsf{P}\big[|X_n| \le Y\big] = 1$ for some $Y \in L_2$ then $\{X_n^2\}$ is UI.

d) T F If $\{X_n\}$ are iid $\mathbb{Z}$-valued then $\mathsf{P}[X_1 < X_2 < \cdots < X_n] < 1/n!$

e) T F If σ-algebras $\{\mathcal{F}_j\}$ are independent, and π-systems $\mathcal{P}_j \subset \mathcal{F}_j$, then the σ-algebras $\{\mathcal{G}_j := \sigma(\mathcal{P}_j)\}$ are independent too.

f) T F If X, Y are independent and $Z = g(Y)$ for some Borel function $g : \mathbb{R} \to \mathbb{R}$, then X, Z are independent.

g) T F If X takes only integer values then the characteristic function $\phi(\omega) := \mathsf{E}\exp(i\omega X)$ is 2π-periodic.

h) T F The characteristic function $\phi(\omega) := \mathsf{E}\exp(i\omega X)$ is real-valued if and only if X and $-X$ have the same distribution.

i) T F If $A, B \in \mathcal{F}$ are disjoint events then $X := \mathbf{1}_A$ and $Y := \mathbf{1}_B$ are independent random variables.

j) T F If $X_n \Rightarrow \mathsf{No}(1, 1)$ and $S_n := \sum_{i \le n} X_i$ then $S_n/n \to 1$ $(pr.)$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	