Sta 711: Homework #3

Random variables

1. Let \((\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \mathcal{B}, \lambda)\) for Lebesgue measure \(\lambda\) on the Borel sets of the unit interval. For \(\omega \in \Omega\) define:

\[
X_1(\omega) \equiv \min(\omega, 0.6) \quad X_2(\omega) \equiv 1_{(0,1/3]}(\omega) \quad X_3(\omega) \equiv \sqrt{\omega}
\]

Plot each of the CDFs \(F_k(x) \equiv \mathbb{P}[X_k \leq x], x \in \mathbb{R}\), and describe explicitly the \(\sigma\)-algebras \(\mathcal{F}_k := \sigma(X_k)\).

2. Let \(X\) be a random variable with CDF \(F(x) := \mathbb{P}(X \leq x)\). Set \(Y \equiv F(X)\). If \(X\) has a continuous distribution (i.e., if \(F\) is a continuous function), show that \(Y\) is a random variable and that \(Y\) has a uniform distribution on \([0,1]\). Warning: \(F(x)\) may not be strictly increasing, and so may not be one-to-one.

3. A random variable \(Y\) is real-valued if \(Y(\omega) \in \mathbb{R}\) for every \(\omega \in \Omega\), and is bounded if there is a fixed finite number \(0 \leq B < \infty\) for which \(|Y(\omega)| \leq B\) for all \(\omega \in \Omega\). Give an example of a real-valued random variable \(X\) that is not bounded.

4. Let \(X\) be a real valued random variable (so \(\mathbb{P}(|X| < \infty) = 1\)) with CDF \(F(x)\). For each \(\epsilon > 0\), construct a bounded random variable \(Y_\epsilon\) such that

\[
\mathbb{P}(X \neq Y_\epsilon) < \epsilon.
\]

Measurable functions

5. Let \(\Omega = \mathbb{R}\). Show that \(\mathcal{S} := \{\emptyset, (-\infty, 0], (0, \infty), \Omega\}\) is a \(\sigma\)-algebra. Describe all functions \(f : \Omega \to \mathbb{R}\) that are \(\mathcal{S}\setminus\mathcal{B}\)-measurable.

6. If \(X\) is a real-valued random variable on any probability space \((\Omega, \mathcal{F}, \mathbb{P})\), then show that \(|X|\) is also a random variable. Show by an example that the converse need not be true (Hint: A finite \(\Omega\) will suffice)

7. Let \(\Omega = \mathbb{R}\), and let \(\mathcal{S}_0 \equiv \{\emptyset, \Omega\}\) be the trivial \(\sigma\)-algebra. Find all measurable functions \(X : (\Omega, \mathcal{S}_0) \to (\mathbb{R}, \mathcal{B})\).

8. Let \(\mathcal{F}_X := \sigma(X)\) be the \(\sigma\)-algebra generated by the function \(X(\omega) = \omega^2\) on \(\Omega = \mathbb{R}\). Is the set \(A = (-\infty, 0]\) in \(\mathcal{F}_X\)? How about \(B = [-4, 4]\)? Why?

9. Let \(\{X_n, n \geq 0\}\) be real-valued random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) that satisfy

\[
\limsup_{n \to \infty} X_n(\omega) = +\infty
\]
for every $\omega \in \Omega$, and let $B < \infty$ be a real number. Prove that the integer-valued quantity
$$\tau(\omega) \equiv \inf \{n \geq 0 : X_n(\omega) \geq B\}$$
is a random variable. 
Extra credit: Prove that $X_\omega$ is also a random variable.

Random Variables and $\sigma$-Algebras

10. All parts of this problem concern the same probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = (0, 1]$, $\mathcal{F} = B(\Omega)$ the Borel sets, and $P = \lambda$ Lebesgue measure. Let $\delta_n(\omega)$ be the $n$th bit in the binary expansion of $\omega$, given by
$$\delta_n(\omega) := [1 + 2^n \omega] \pmod{2}$$
where $[x]$ is the least integer $\geq x$, and set
$$\mathcal{F}_n := \sigma \{\delta_1, \ldots, \delta_n\} = \sigma \{(0, j/2^n] : j = 0, \ldots, 2^n\}.$$

(a) Find a single real-valued random variable $X$ on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_3 = \sigma(X)$.

(b) True or False: If $Y$ is any other random variable on $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_3 = \sigma(Y)$, then $Y = g(X)$ for some Borel measurable function $g : \mathbb{R} \to \mathbb{R}$. Give a proof or counter-example.

(c) Let $Z$ be a random variable on $(\Omega, \mathcal{F}, P)$ for which $\mathcal{F} = \sigma(Z)$ (recall $\mathcal{F} = B(\Omega)$, $\Omega = (0,1]$, and $P = \lambda$). True or false: For each $\omega_1 \neq \omega_2$, necessarily $Z(\omega_1) \neq Z(\omega_2)$. Explain.