Sta 711 : Homework 5

1. Independence.
   (a) Let \( \{B_i\} \) be independent events. For \( n \in \mathbb{N} \) show that
   \[
   \mathbb{P}\left(\bigcup_{i=1}^{n} B_i\right) = 1 - \prod_{i=1}^{n} [1 - \mathbb{P}(B_i)] \geq 1 - \exp\left(-\sum_{i=1}^{n} \mathbb{P}(B_i)\right)
   \]

   (b) If \( \{A_n, n \in \mathbb{N}\} \) is a sequence of events such that \( \mathbb{P}[A_n] = 1/3 \) for each \( n \) and
       \[
       (\forall n \neq m \in \mathbb{N}) \quad \mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n) \mathbb{P}(A_m),
       \]
   does it follow that the events \( \{A_n\} \) are independent? Give a proof or counterexample. Note \( 1/3 \neq 1/2 \).

   (c) Show that a random variable \( Y \) is independent of itself if and only if, for some constant \( c \in \mathbb{R} \), \( \mathbb{P}[Y = c] = 1 \).

   Let \( f : \mathbb{R} \to \mathbb{R} \) be Borel measurable, and \( X \) any random variable. Can \( Y := f(X) \) and \( X \) be independent? Explain your answer.

   (d) Give an example to show that an event \( A \in \mathcal{F} \) may be independent of each \( B \) in some collection \( \mathcal{C} \subset \mathcal{F} \) of events, but not independent of \( \sigma(\mathcal{C}) \). Prove this is impossible if \( \mathcal{C} \) is a \( \pi \)-system (i.e., in that case \( A \) must be independent of \( \sigma(\mathcal{C}) \)).

   (e) Give a simple example to show that two random variables on the same space \( (\Omega, \mathcal{F}) \) may be independent according to one probability measure \( \mathbb{P}_1 \) but dependent with respect to another \( \mathbb{P}_2 \).

   (a) Let \( \{X_n\} \) be a sequence of Bernoulli random variables with
       \[
       \mathbb{P}(X_n = 1) = n^{-p} \quad \mathbb{P}(X_n = 0) = 1 - n^{-p}
       \]
   for some \( p > 0 \). For \( p = 2 \) show that the partial sum
       \[
       S_n := \sum_{k=1}^{n} X_k
       \]
   converges almost-surely, whether or not the \( \{X_n\} \) are independent. If the \( \{X_n\} \) are independent, for which \( p > 0 \), does \( S_n \) converge? Why?

   (b) Let \( \{X_n\} \) be an iid sequence of random variables with a non-degenerate distribution (i.e., for some \( B \in \mathcal{B} \), \( 0 < \mathbb{P}[X_n \in B] < 1 \)). Show that
       \[
       \mathbb{P}[\omega : X_n(\omega) \text{ converges}] = 0
       \]
(c) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables \( \{X_n\} \), there exist constants \( c_n \to \infty \) such that

\[
P \left( \lim_{n \to \infty} \frac{X_n}{c_n} = 0 \right) = 1.
\]

Give a careful description of how you choose \( c_n \) (it will depend on the distributions of the \( X_n \)). Find a suitable sequence \( \{c_n\} \) explicitly for an iid sequence \( \{X_n\} \sim \text{Ex}(1) \) of unit-rate exponentially-distributed random variables to ensure that \( X_n/c_n \to 0 \) almost surely.


(a) Suppose \( \{A_n, n \in \mathbb{N}\} \) are independent events satisfying \( P(A_n) < 1, \forall n \in \mathbb{N} \). Show that \( P(\bigcup_{n=1}^{\infty} A_n) = 1 \) if and only if \( P(A_n \text{ i.o.}) = 1 \) (“i.o.” means “infinitely often”, so the question concerns \( \lim \sup A_n \)). Give an example to show that the condition \( P(A_n) < 1 \) cannot be dropped.

(b) Suppose \( \{A_n\} \) is a sequence of events. If \( P(A_n) \to 1 \) as \( n \to \infty \), prove that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( P(\bigcap_{k} A_{n_k}) > 0 \).

(c) Let \( A_n \) be a sequence of events. If there exists \( \epsilon > 0 \) such that \( P(A_n) \geq \epsilon \) for all \( n \in \mathbb{N} \), does it follow that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( P(\bigcap_{k} A_{n_k}) > 0 \)? Why or why not?

(d) Let \( \{X_n\} \) be non-negative iid random variables, with tail \( \sigma \)-field

\[
\mathcal{T} \equiv \bigcap_{n} \mathcal{F}_n^\prime, \quad \mathcal{F}_n^\prime \equiv \sigma \{X_m : m > n\}
\]

Is the event

\[
E = \{\text{There exists } \epsilon > 0 \text{ such that } X_n > n \epsilon \text{ for infinitely-many } n\}
\]

\[
= \bigcup_{\epsilon > 0} \bigcap_{n \geq 1} \bigcup_{m \geq n} \{\omega : X_m(\omega) > m \epsilon\}
\]

in \( \mathcal{T} \)? Prove or disprove it.

Express the probability \( P[E] \) in terms of the random variables’ common distribution—for example, using their common CDF \( F(x) \equiv P[X_n \leq x] \) or moments \( E[|X_n|^p] \) for some \( p > 0 \).