Sta 711 : Homework 6

Almost-sure and In-probability Convergence

1. Let \( \{X_n\} \) be a monotonically increasing sequence of RVs such that \( X_n \to X \) in probability (pr). Show that \( X_n \to X \) almost surely (a.s.)

2. Let \( \{X_n\} \) be any sequence of RVs. Show that \( X_n \to X \) a.s. if and only if
   \[
   \sup_{k \geq n} |X_k - X| \to 0 \quad \text{pr.}
   \]

3. Let \( \{X_n\} \) be an arbitrary sequence of RVs and set \( S_n := \sum_{i=1}^{n} X_i \). Show that \( X_n \to 0 \) a.s. implies that \( S_n/n \to 0 \) a.s.

4. Let \( \{X_n\} \subset L_2 \) be independent and identically distributed. For each \( \delta > 0 \) show that \( n P \{ |X_1| > \delta \sqrt{n} \} \to 0 \). Use this to show that the maximum \( \max_{i=1}^{n} |X_i|/\sqrt{n} \to 0 \) pr. Thus, the maximum of \( n \) iid \( L_2 \) random variables grows slower than \( \sqrt{n} \).

5. For random variables \( X, Y \) define
   \[
   \rho(X, Y) := E \left\{ \frac{|X - Y|}{1 + |X - Y|} \right\}
   \]
   The function \( \rho \) is a metric (you do not have to prove that), i.e., it’s non-negative, symmetric, satisfies the triangle inequality, and vanishes if and only if \( X = Y \) a.s. Show that \( X_n \to X \) pr. if and only if \( \rho(X_n, X) \to 0 \). Thus, convergence in probability is metrizable.\(^1\)

\( L_p \) Convergence

6. Let \( \{X_n\} \subset L_1(\Omega, \mathcal{F}, P) \) be a sequence of positive RVs that converge in probability to \( X \in L_1(\Omega, \mathcal{F}, P) \). Show that \( E(X_n) \to E(X) \) if and only if \( X_n \to X \) in \( L_1 \).

7. Find a sequence of RVs \( \{X_n\} \subset L_2 \) which converge in \( L_1 \) but not in \( L_2 \).

8. Let \( (\Omega, \mathcal{F}, P) := ((0, 1), \mathcal{B}, \lambda) \) be the unit interval with Borel sets and Lebesgue measure and define \( X_n(\omega) := \omega^n \) for \( n \in \mathbb{N}, \omega \in \Omega \). For what \( p \in [1, \infty] \), does the sequence \( \{X_n\} \) converge in \( L_p \)? To what limit? Explain your answer.

9. Verify Hölder’s inequality for \( p = 1, q = \infty \) and all random variables \( X, Y \):
   \[
   E|XY| \leq ||X||_1 ||Y||_\infty
   \]
   where \( ||Y||_\infty := \sup\{c < \infty : P\{|Y| > c\} > 0\} \).

10. Verify Minkowski’s inequality for \( p = \infty \) and all random variables \( X, Y \):
    \[
    ||X + Y||_\infty \leq ||X||_\infty + ||Y||_\infty
    \]

\(^1\)Many other metrics would work too— like \( E(|X - Y| \wedge 1) \) or \( \inf\{\epsilon > 0 : P(|X - Y| > \epsilon) \leq \epsilon\} \).